Math 2210-1
Fri Oct 22
§ 13.7-13.9
Chain rule
directional derivative & gradient
Lagrange multipliers

- State the chain rule for derivative of $\vec{G}(\vec{F}(\vec{x}))$

- State the important special case $f(\vec{r}(t))$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\vec{r}"] \arrow[rr, bend right, "f\circ\vec{r}"'] & \mathbb{R}^n \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

- Define directional derivative, $D_{\vec{u}} f(\vec{x})$
and give computational formula.
Interpret geometrically.

Math 2210-3
Illustrating the meaning of the gradient via level curves
Friday, October 22, 2004

Example

$g(x,y) = 2x(1-x^2-y^2)$

$-1 \le x \le 1$
$-1 \le y \le 1$

so $g(x,y) = 2x - 2x^3 - 2xy^2$

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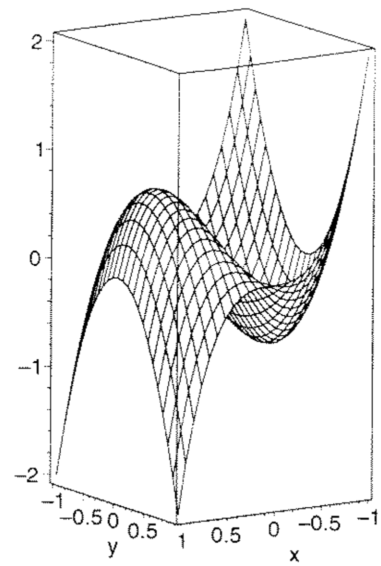
> with(plots):
> g:=(x,y)->2*x*(1-x^2-y^2);
```

$$g := (x, y) \to 2\,x\,(1 - x^2 - y^2)$$

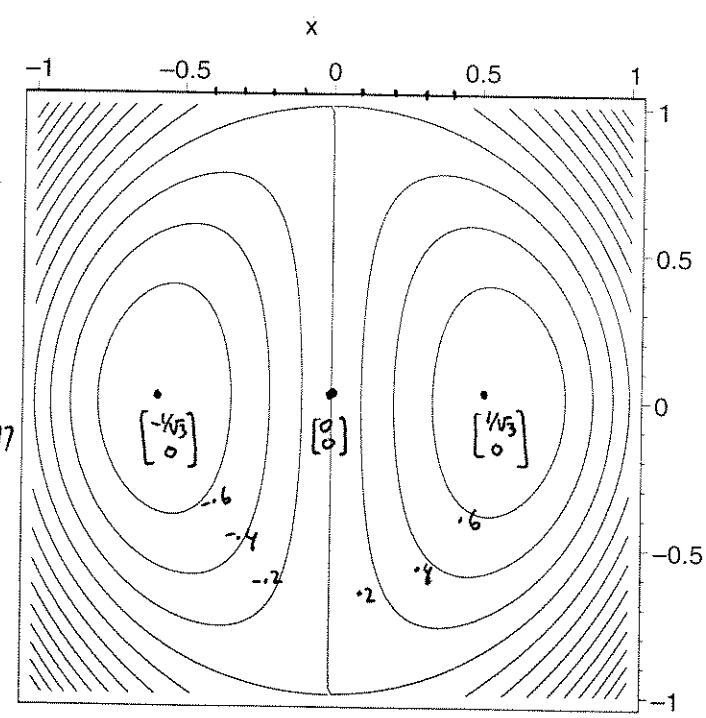
```
> plot3d(g(x,y),x=-1..1,y=-1..1,
  contours=20,axes=boxed,
  orientation=[57,78],
  color=white,
  scaling=constrained);
```

graph
$z = g(x,y)$



Given that the level curves of g(x,y) ~~above~~ below are plotted in (height) increments of 0.2, estimate the gradient vector of g at the origin, just using the level curve picture (and the 3-d graph of g). Then compare with the actual value of the gradient, which you compute from the formula for g.

local min
$f\begin{pmatrix}-1/\sqrt{3}\\0\end{pmatrix} \approx -.77$



y

local max
$f\begin{pmatrix}1/\sqrt{3}\\0\end{pmatrix} \approx .77$

Theorem Let $f : \mathbb{R}^n \to \mathbb{R}$ differentiable.

Consider the level set in $\mathbb{R}^n$,

$$f(\vec{x}) = c \qquad (c \text{ fixed scalar})$$

Let $\vec{P}$ be in this level set.

Then $\nabla f(\vec{P})$ is $\perp$ to the level set (i.e. to the tangent plane or tangent line of the level set.)

pf Let $\vec{r}(t)$ a curve to the level set,

i.e. $f(\vec{r}(t)) \equiv c$,

with $\vec{r}(0) = \vec{P}$

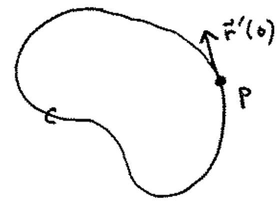
Then $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \equiv 0$

at $t=0$, $\boxed{\nabla f(\vec{P}) \cdot \vec{r}\,'(0) = 0.}$

Since we could pick $\vec{r}\,'(0)$ to be any tangent vector of the level set at $\vec{P}$, deduce $\nabla f(\vec{P}) \perp$ level set.

∎

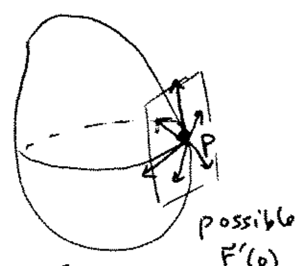
$n=2$ domain



$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ s.t. } f\begin{pmatrix} x \\ y \end{pmatrix} = c \right\}$

level curve.

$n=3$ domain



$f = c$.

<u>Example</u>: Find the tangent plane to the sphere $x^2+y^2+z^2=9$

@ $\begin{bmatrix}1\\2\\2\end{bmatrix}=P$

3 ways!

(1) near P, the sphere is a graph $z=f(x,y)=+\sqrt{9-x^2-y^2}$

so we can use the method of § 13.4 (p. 916):

$f_x=\dfrac{-x}{\sqrt{9-x^2-y^2}}$ @ $\begin{bmatrix}1\\2\end{bmatrix}$ $f_x=-\frac{1}{2}$

$f_y=\dfrac{-y}{\sqrt{9-x^2-y^2}}$ $f_y=-\frac{2}{2}=-1$

$$\boxed{z=2+(-\tfrac{1}{2})(x-1)-1(y-2)}$$

or

$$\tfrac{1}{2}x+y+z=\tfrac{9}{2}$$

$$\boxed{x+2y+2z=9}$$

(2) We can find $\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial y}=\frac{\partial f}{\partial y}$ by implicit differentiation, as in example 11 § 13.7 (p. 952) and as in HW!

(3) We can use the theorem we just proved, since the sphere is a level surface!

Lagrange multipliers

for constrained optimization problems.

If we want to maximize (or minimize)

$f(\vec{x})$

subject to the constraint

$g(\vec{x}) = c$ (constant)

Then we are really extremizing f on a level set ($\overset{n=2}{\text{level curve}}$, $\overset{n=3}{\text{level surface}}$).

At such an extreme point $\vec{P}$ in the level set,

for any curve $\vec{r}(t)$ into the level set
$\vec{r}(0) = \vec{P}$

then $f \circ \vec{r}(t)$ will have a local max or min at $t=0$

so $\nabla f(P) \cdot \vec{r}\,'(0) = 0$ by chain rule.

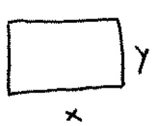
i.e. $\nabla f(\vec{P})$ <u>also</u> $\perp$ to level set $g = c$.

i.e. $\nabla f(\vec{P}) \parallel \nabla g(\vec{P})$

$$\boxed{\nabla f(\vec{P}) = \lambda \nabla g(\vec{P})}$$ λ = "Lagrange multiplier"

<u>examples</u>

1210:



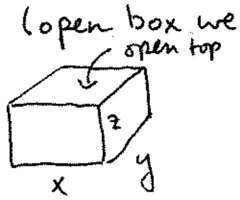
Minimize perim $= 2x + 2y$ ← $f(x,y)$

subject to $xy = C$ (constant area). ← $g(x,y)$.

$\nabla f = \lambda \nabla g$

$\langle 2, 2 \rangle = \lambda \langle y, x \rangle$ $\quad$ $\begin{matrix} 2 = \lambda y \\ 2 = \lambda x \end{matrix} \Rightarrow \boxed{x = y}$ $(= \sqrt{c})$.

2210: (open box we did by eliminating constraint, page 4 10/13):



$xyz = 4$ ← $g(x,y,z)$

minimize $xy + 2xz + 2yz$ ← $f(x,y,z)$

$$\nabla f = \langle y+2z,\ x+2z,\ 2x+2y \rangle = \lambda \nabla g = \lambda \langle yz, xz, xy \rangle$$

$$\left.\begin{aligned} \frac{y+2z}{yz} &= \lambda = \frac{1}{z} + \frac{2}{y} \\ \frac{x+2z}{xz} &= \lambda = \frac{1}{z} + \frac{2}{x} \end{aligned}\right\} \Rightarrow x = y$$

$$\frac{2x+2y}{xy} = \lambda = \frac{2}{y} + \frac{2}{x} \Rightarrow \lambda = \frac{4}{x} = \frac{1}{z} + \frac{2}{x} \Rightarrow \frac{2}{x} = \frac{1}{z} \Rightarrow z = \tfrac{1}{2}x \ !$$