Math 2210-1
Wed 10/20

§ 13.6–13.8 cont'd.
differentiability
chain rule

Recall,

$\vec{F}: D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$

is differentiable at $\vec{x} \in D$ means
good affine approximation, precisely

$$\vec{F}(\vec{x}+\vec{h}) = \vec{F}(\vec{x}) + \vec{F}'(x)\vec{h} + \text{error}$$

where $\frac{|\text{error}|}{|\vec{h}|} \to 0$ as $\vec{h} \to 0$.

Here $[\vec{F}'(x)]_{m\times n}$ is the derivative matrix of $\vec{F}$,
$\text{entry}_{ij} = \frac{\partial F_i}{\partial x_j}(\vec{x})$,

and is the only matrix "C" for which one could have such an affine approx formula.

Chain rule :

$\vec{F}: D \longrightarrow \mathbb{R}^m$, $D \subset \mathbb{R}^n$
$G: \mathcal{D} \longrightarrow \mathbb{R}^k$, $\mathcal{D} \subset \mathbb{R}^m$

If $\vec{F}$ is diff'ble at $\vec{x}$
and $\vec{G}$ is diff'ble at $\vec{F}(\vec{x})$
then $G \circ F$ is diff'ble at $\vec{x}$ and

$$[(G\circ F)'(x)] = [G'(F(x))][F'(x)]$$

Just like 1210 chain rule, except matrixes & matrix products!

Idea of proof is easy – for details of proof when partial derivs are continuous,
see text page 952–953

$$\vec{F}(\vec{x}+\vec{h}) \approx \vec{F}(\vec{x}) + [\vec{F}'(x)]\vec{h}$$
$$\vec{G}(\vec{y}+\vec{k}) \approx \vec{G}(y) + [\vec{G}'(y)]\vec{k}$$

"y" "k")

$$\text{so } G(F(x+h)) \approx G[\underbrace{F(x)}_{\text{"y"}} + \underbrace{F'(x)h}_{\text{"k"}}]$$
$$\approx G(F(x)) + [G'(F(x))][F'(x)h]$$
$$= G(F(x)) + ([G'(F(x))][F'(x)])h$$

got tired of vector arrows, but they're still there!

is affine approx for $G\circ F$, so

is the derivative matrix!

Examples

1. $\vec{r}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ $\qquad$ $\begin{tikzcd} \mathbb{R} \arrow[r, "\vec{r}"] \arrow[rr, bend right, "f\circ\vec{r}"'] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$

   $f\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2$

$$\frac{d}{dt} f(\vec{r}(t)) = [f_x, f_y]\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

$$= [2x, 2y]\begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

$$= [2t, 2t^2]\begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

$$\boxed{= 2t + 4t^3}$$

check:
$f(\vec{r}(t)) = t^2 + t^4$
$D_t(\%) = 2t + 4t^3$!

2. In general

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{\vec{r}(t)} & \mathbb{R}^n & \xrightarrow{f} & \mathbb{R} \\ t & & \vec{x} & & \end{array}$$

evaluated at $\vec{r}(t)$

$$\frac{d}{dt} f(\vec{r}(t)) = [\ f'(\vec{r}(t))\ ]\begin{bmatrix} \vec{r}'(t) \end{bmatrix} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n}\right]\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

$$\boxed{\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)}$$

most important special case of the chain rule.

$$\boxed{= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \dots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}}$$

$\left(\frac{\partial f}{\partial x_j}\ \text{evaluated at}\ \vec{x} = \vec{r}(t)\right)$.

3. A special case of ②, is a re-derivation of the directional derivative formula

$$D_{\vec{u}} f(\vec{x}) := \lim_{h\to 0} \frac{f(\vec{x}+h\vec{u}) - f(\vec{x})}{h} \quad \text{also} = \frac{d}{dt}\Big(f(\overbrace{\vec{x}+t\vec{u}}^{\vec{r}(t)})\Big)\Big|_{t=0}$$

$(|\vec{u}|=1)$

$$\boxed{D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}}$$

by ②, since
$\vec{r}(0) = \vec{x}$
$\vec{r}'(0) = \vec{u}$.

example

The air temperature near a lightbulb outside is given by

$$f(x,y,z) = 32 + 68\, e^{-x^2-y^2-z^2}$$

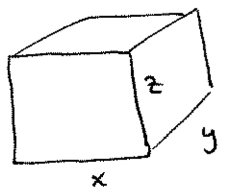
degrees Farenhat
distance in m.

- Compute $\nabla f(x,y,z)$
  describe geometrically.

- What are the level surfaces of temperature.
  (describe geometrically)
  observe they are $\perp$ to the gradient vector field!

- a moth is spiraling towards the light with speed $.3$ m/s
  at time $t=0$ it is at $\begin{bmatrix}1\\0\\0\end{bmatrix}$
  and moving in direction $\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\end{bmatrix}$.
  How fast is the moth's outside temp changing?

<u>example</u> (related rates)



Melting ice block.

top face melting $.1$ cm/min
side faces @ $.05$ cm/min
bottom face @ $.02$ cm/min

When $x = y = 20$ cm
$z = 15$ cm

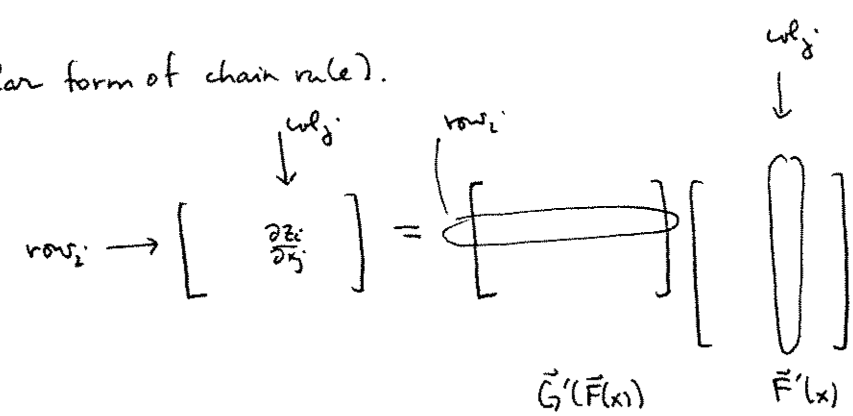
How fast is volume changing? $\frac{dV}{dt}$
cm^3/min.

<u>example</u> (matrix vs. scalar form of chain rule).

$\vec{F}(\vec{x}) = \vec{y}$

$G(\vec{y}) = z$

$\frac{\partial z_i}{\partial x_j} = \text{entry}_{ij}([g \circ f'(x)])$



$$\boxed{\frac{\partial z_i}{\partial x_j} = \frac{\partial z_i}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \frac{\partial z_i}{\partial y_2}\frac{\partial y_2}{\partial x_j} + \cdots + \frac{\partial z_i}{\partial y_m}\frac{\partial y_m}{\partial x_j}}$$

= "sum of (derivs of z_i wrt <u>its variables</u> times derivs of those variables wrt x_j)".

<u>example</u>

$\frac{\partial p}{\partial u}$ where

$p = f(x,y,z)$

$x = x(u,v)$
$y = y(u,v)$
$z = z(u,v)$

$$\begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to p$$

Homework: (For Wed 10/27).

13.7 1 3 5 7, $\underbrace{9, 13}$

└ also exhibit the product (unmultiplied out) of the derivative matrices as they appear in chain rule

23, 29, 33, 35, 40

13.8 7 11, 12, 23, 44, 45, 49

13.9 3 11 19 31 17

And

I for $r > 0$, $-\pi/2 < \theta < \pi/2$ Define $\vec{F}\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

for $x > 0$ define $\vec{G}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{x^2+y^2} \\ \arctan y/x \end{bmatrix}$

(a) Verify that $\vec{F}$ & $\vec{G}$ are inverse functions, i.e.

$$G \circ F \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} r \\ \theta \end{bmatrix}$$

$$F \circ G \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) Explain why this means that the derivative matrices of $G \circ F$ and $F \circ G$ are $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) Thus the chain rule implies

$$\left[G'\left(F\begin{pmatrix} r \\ \theta \end{pmatrix}\right)\right]\left[F'\begin{pmatrix} r \\ \theta \end{pmatrix}\right] = I$$

verify this

II Use Lagrange multipliers to show that the cylinder with minimum surface area, for fixed volume, satisfies $2r = h$ (i.e. diam = h)

sides + top + bottom