Math 2210-1
Fri 10/15

§13.6 Affine approx, differentials,
derivative matrices.

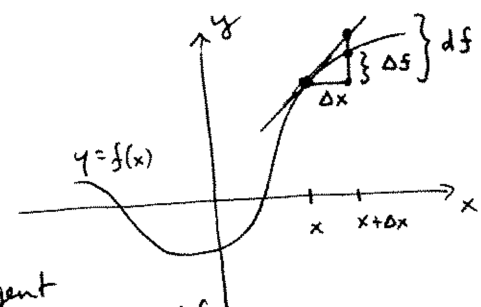
Recall Calc 1210:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

so $\dfrac{f(x+\Delta x) - f(x)}{\Delta x} - f'(x) = \varepsilon$, where $\lim\limits_{\Delta x \to 0} \varepsilon = 0$.

i.e. $$\underbrace{f(x+\Delta x) - f(x)}_{\text{"}\Delta f\text{"}} = \underbrace{f'(x)\,\Delta x}_{\text{"}df\text{"}} + \underbrace{\varepsilon\,\Delta x}_{\text{higher order error}}$$

(or Δy if $y = f(x)$).



differential
df was the tangent
line approx to exact change Δf.

example: estimate
$\sqrt{10}$ using $\sqrt{9} = 3$.

$f(x) = x^{1/2}$ $\qquad f'(x) = \frac{1}{2} x^{-1/2}$

$\Delta x = 1$

$x = 9$

$$f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x$$
$$= 3 + \frac{1}{2}\frac{1}{3}\cdot 1 \cong 3.17$$

(actual value 3.162...)

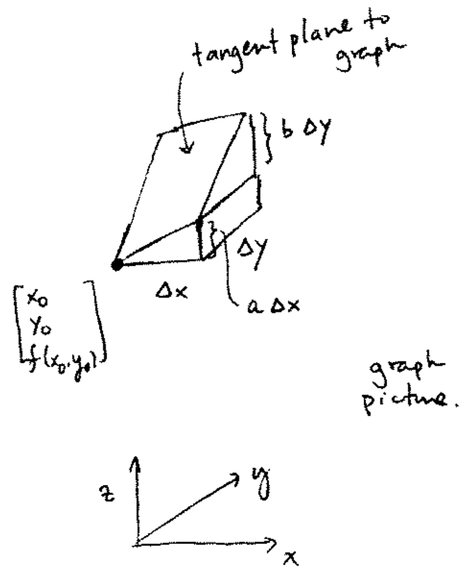
What is the analogous approximation for a function of several variables?

e.g. $f(x,y)$.

Geometric idea: consider the graph of $z = f(x,y)$ and the tangent plane approximation

$$z = f(x_0,y_0) + a(x-x_0) + b(y-y_0)$$

$$a = f_x(x_0,y_0)$$
$$b = f_y(x_0,y_0)$$

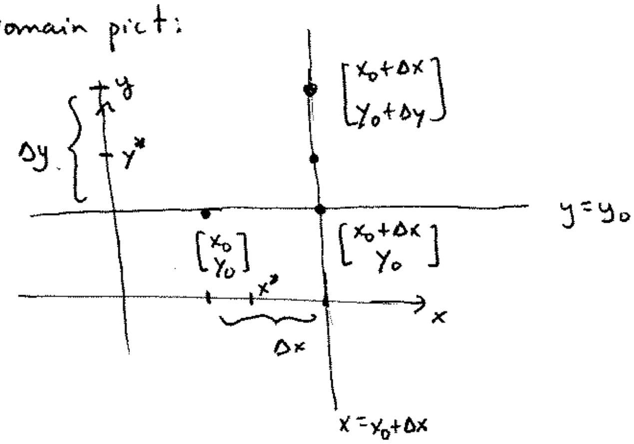


$$\begin{aligned}\Delta f &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \\ &\approx a\,\Delta x + b\,\Delta y \\ &= \underbrace{\frac{\partial f}{\partial x}(\cdot)\,\Delta x + \frac{\partial f}{\partial y}(\cdot)\,\Delta y.}_{:=\, df}\end{aligned}$$

More precise explanation:

In fact, if the partial derivs of f are continuous we can prove the affine approx. formula with the Mean value thm from Calculus.

Domain pict:



$$\begin{aligned}\Delta f &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \\ &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0) \\ &\quad + f(x_0+\Delta x, y_0) - f(x_0,y_0) \\ &= \frac{\partial f}{\partial y}(x_0+\Delta x, y^*)\,\Delta y \\ &\quad + \frac{\partial f}{\partial x}(x^*, y_0)\,\Delta x\end{aligned}$$

Mean-value thm!

When $\Delta x, \Delta y$ small, $y^* \approx y_0$, $x^* \approx x_0$, so

$$\boxed{\Delta f = \frac{\partial f}{\partial x}(x_0,y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y}$$

$$\varepsilon_1 = \frac{\partial f}{\partial x}(x^*,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)$$

$$\varepsilon_2 = \frac{\partial f}{\partial y}(x_0+\Delta x, y^*) - \frac{\partial f}{\partial y}(x_0,y_0)$$

example Approximate $\sqrt{2(2.02)^3 + (2.97)^2}$ with differentials

$$f(x,y) = \sqrt{2x^3 + y^2} = (2x^3+y^2)^{\frac{1}{2}}$$

$$f(2,3) = \sqrt{16+9} = 5.$$

$\Delta x = .02$
$\Delta y = -.03$

$$f_x = \tfrac{1}{2}(2x^3+y^2)^{-\frac{1}{2}} \cdot 6x^2$$

$$f_y = \tfrac{1}{2}(2x^3+y^2)^{-\frac{1}{2}}\, 2y$$

$$f_x(2,3) = \tfrac{3}{5} \cdot 4 = \tfrac{12}{5} = 2.4$$

$$f_y(2,3) = \tfrac{1}{10} \cdot 6 = .6$$

$$df = f_x\,\Delta x + f_y\,\Delta y$$

$$= (2.4)(.02) + (.6)(-.03)$$

$$= .048 - .018$$

$$= .03$$

So $f(2.02, 2.97) \approx 5 + .03$
$= 5.03$

(actual value ≈ 5.0305)

evaluated at base point

Notice

$$\Delta f \approx df = [f_x,\ f_y]\begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix}$$

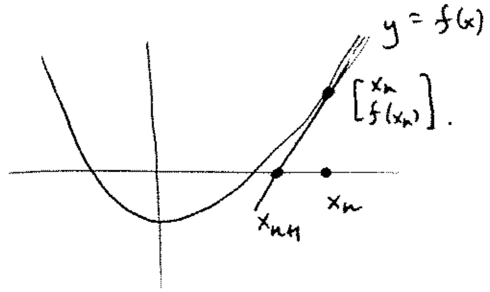
So if $\vec{F}\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y)\end{bmatrix}$, $\vec{F}: D \to \mathbb{R}^2$, $D \subset \mathbb{R}^2$

We get approx formula

$$\Delta\vec{F} = \begin{bmatrix}\Delta f \\ \Delta g\end{bmatrix} \approx \begin{bmatrix} df \\ dg\end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y\end{bmatrix}\begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix}$$

the derivative matrix of $\vec{F}$!

Recall Newton's method. (for finding roots)



$f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x$

$\downarrow$

set $= 0$, solve for Δx:

$0 = f(x) + f'(x)\,\Delta x$

$\Delta x = -\dfrac{f(x)}{f'(x)}$

yields

Newton iteration:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

<u>example</u>

Affine approximations and Newton's Method
Math 2210-1, October 16, 2004

In calculus you used the tangent-line approximation to the graph y=f(x) of a function f, in order to find roots of f. This iteration procedure goes back to Newton. Recall, the iteration scheme is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{D_x(f(x_n))}$$

Example: the square root of 2: We use

$$f(x) = x^2 - 2$$

```
> Digits:=30:  #number of digits to use in computations
> f:=x->x^2-2: #define function
  x0:=1.5:  #initial guess
  z:=x0:  #initialize
  for i from 1 to 5 do
  x1:=z: #dummy variable
  z:=x1-f(x1)/(2*x1); #Newton iteration
  print(z);
  od:
```

1.41666666666666666666666666667

1.41421568627450980392156862745

1.41421356237468991062629557889

1.41421356237309504880168962350

1.41421356237309504880168872421

```
> sqrt(2.0);  #actual value
```

1.41421356237309504880168872421

Pretty fast convergence, wouldn't you say?!

Amazingly, Newton's method works in the multivariable setting too

Example 7 p. 940

$$\vec{F}(\vec{x}+\Delta\vec{x}) \approx \vec{F}(\vec{x}) + \underbrace{\vec{F}'(x)}_{n\times n \text{ deriv matrix}} \Delta\vec{x}$$

set $= 0$, solve for $\Delta\vec{x}$:

$$\vec{F}'(x)\,\Delta\vec{x} = -\vec{F}(\vec{x})$$

$$\Delta\vec{x} = -[\vec{F}'(x)]^{-1}\vec{F}(\vec{x})$$

This is the Newton step!

EXAMPLE 7 Find (accurate to three decimal places) the coordinates of the first-quadrant point P of intersection of the ellipses $x^2+4y^2=13$ and $x^2-2xy+9y^2=17$ (Fig. 13.6.3).

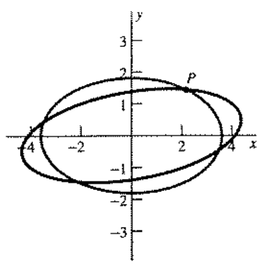


FIGURE 13.6.3 The ellipses $x^2+4y^2=13$ and $x^2-2xy+9y^2=17$ of Example 7.

Solution We need to solve the simultaneous scalar equations

$$f(x,y)=x^2+4y^2-13=0 \quad \text{and} \quad g(x,y)=x^2-2xy+9y^2-17=0.$$

Text

Hence we define the vector-valued function

$$F(\mathbf{x}) = \begin{bmatrix} x^2+4y^2-13 \\ x^2-2xy+9y^2-17 \end{bmatrix}$$

with derivative matrix

$$F'(\mathbf{x}) = \begin{bmatrix} 2x & 8y \\ 2x-2y & -2x+18y \end{bmatrix}.$$

Maple

```
> f:=(x,y)->x^2+4*y^2-13;
  g:=(x,y)->x^2-2*x*y+9*y^2-17;
```

$$f := (x,y) \to x^2+4\,y^2-13$$

$$g := (x,y) \to x^2-2\,x\,y+9\,y^2-17$$

The derivative matrix is given by

```
> with(linalg):
> Derivmat:=(z,w)->subs({x=z,y=w},
  matrix(2,2,[diff(f(x,y),x),diff(f(x,y),y),
  diff(g(x,y),x),diff(g(x,y),y)]));
  Derivmat(x,y); #check!
```

$$Derivmat := (z,w) \to \mathrm{subs}\left(\{x=z, y=w\}, \mathrm{matrix}\left(2, 2, \left[\frac{\partial}{\partial x}\mathrm{f}(x,y), \frac{\partial}{\partial y}\mathrm{f}(x,y), \frac{\partial}{\partial x}\mathrm{g}(x,y), \frac{\partial}{\partial y}\mathrm{g}(x,y)\right]\right)\right)$$

$$\begin{bmatrix} 2\,x & 8\,y \\ 2\,x-2\,y & -2\,x+18\,y \end{bmatrix}$$

```
> s:=2;
  t:=1.5; #initial guess of point [s,t]
```

$$s := 2$$

$$t := 1.5$$

```
> for i from 1 to 5 do
  newvect:=evalm(vector([s,t]) -evalm(Derivmat(s,t)^(-1)&*vector
  ([f(s,t),g(s,t)]))); #Newton step
  s:=newvect[1];
  t:=newvect[2];
  print({s,t});
  od:
>
```

$$\{2.187500000000000000000000000000,\ 1.437500000000000000000000000000\}$$

$$\{2.187322009764035801464605370\text{22},\ 1.433151952807160292921074043\text{94}\}$$

$$\{2.187326102353510282036581868\text{21},\ 1.433143792702436714080527050\text{15}\}$$

$$\{2.187326102376595836188469605\text{68},\ 1.433143792668936080950677579\text{26}\}$$

$$\{2.187326102376595836188889208\text{59},\ 1.433143792668936080950079442\text{37}\}$$

```
> fsolve({f(x,y)=0,g(x,y)=0},{x,y},{x=1..3,y=0..3});
  #Maple knows how to do these things too
>
```

$$\{x = 2.187326102376595836188889208\text{59},\ y = 1.433143792668936080950079442\text{37}\}$$