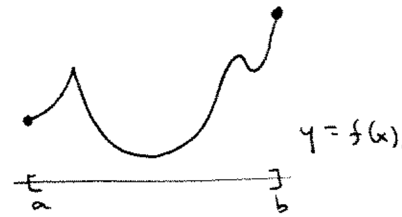
Math 2210-1
Wed Oct 13

Max-min problems § 13.5




Remember in Calc.

If $f$ is continuous on a closed interval $[a,b]$.
Then $f$ attains its extreme values, (max & min), either
(i) at the endpoints (the "boundary" of the interval)
or
(ii) at critical points $x_0$ in $(a,b)$ (in the "interior" of the interval)
↳ either $f'(x_0)=0$ or $f'(x_0)$ DNE.

[Because, if $f'(x_0)$ exists and $\neq 0$ $f$ is inc or dec at $x_0$ so can't have even a local extremum].

The analogus fact is true in n-dim'l domain

$f: D \longrightarrow \mathbb{R}$.
$D \subset \mathbb{R}^n$

If $D$ is a closed domain
↓
$D$ includes its boundary.
and if $D$ is a bounded domain
↓ ~~there are~~ all points in $D$ are at most a fixed finite distance from $\vec{0}$.

and if $f$ is continuous on $D$
Then $f$ attains its extreme values, either
(i) along the boundary of $D$
or
(ii) at critical points $\vec{x}_0$ in $D$
↳ either each $\frac{\partial f}{\partial x_i}(\vec{x}_0)=0$ at $\vec{x}_0$
or
not all partial derivs exist at $\vec{x}_0$

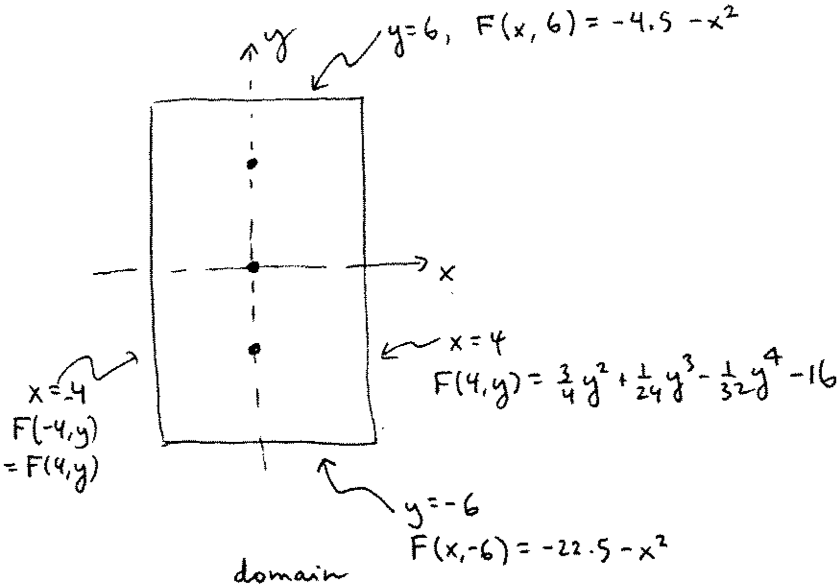
The reasoning is just like in Calc: if some $\frac{\partial f}{\partial x_i}(\vec{x}_0) \neq 0$
then $f$ is increasing or decreasing in the $x_i$ direction, at $\vec{x}_0$.

Example

Let $D$ = rectangle, $-4 \le x \le 4$
$-6 \le y \le 6$

$$F(x,y) = \frac{3}{4}y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4 - x^2$$

Find the extreme values of $F$, by finding all critical points and analyzing the behavior of $F$ along the boundary (perimeter) of the rectangle.




Critical points:

$F_x = -2x = 0 \quad \rightarrow$ so $x = 0$

$$F_y = \frac{3}{2}y + \frac{1}{8}y^2 - \frac{1}{8}y^3 = 0$$
$$= -\frac{1}{8}\left[y^3 - y^2 - 12y\right]$$
$$= -\frac{1}{8}y(y^2 - y - 12)$$
$$= -\frac{1}{8}y(y-4)(y+3) \quad \rightarrow \text{ so } y = 0, 4, -3$$

$$\boxed{(0,0), (0,4), (0,-3)}$$

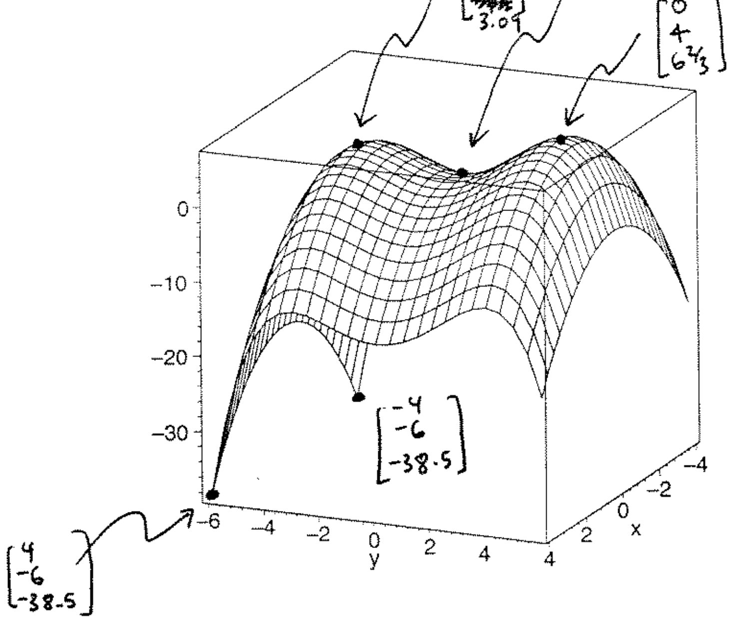
$F(0,0) = 0$
$F(0,4) = 6\frac{2}{3}$
$F(0,-3) \cong 3.09$

Looking along the boundary:

we can approach this systematically, and it helps to use technology (next page)

Math 2210-1 October 13
Pictures for extrema discussion
Example 3, section 13.5

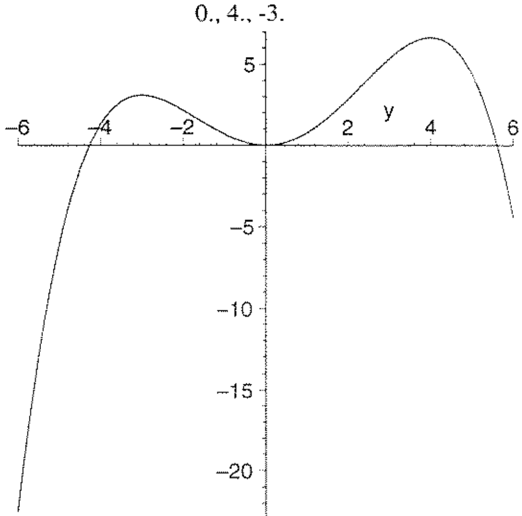
```
> with(plots):
Warning, the name changecoords has been redefined

> with(plots):
> F:=(x,y)->.75*y^2+1/24*y^3-1/32*y^4-x^2:
> plot3d(F(x,y),x=-4..4,y=-6..6,
  axes=boxed,color=white);
```




```
> f:=y->.75*y^2+1/24*y^3-1/32*y^4;
  solve(diff(f(y),y)=0,y);
  plot(.75*y^2+1/24*y^3-1/32*y^4,y=-6..6,color=black);
```

$$f := y \to 0.75\, y^2 + \frac{1}{24} y^3 - \frac{1}{32} y^4$$

0., 4., -3.




```
> f(4);f(-3);f(0);f(6);f(-6);
```

6.66666667

3.093750000

0.

-4.50000000

-22.50000000

```
> F(4,-6);F(-4,-6);  #minimum values on the rectangle
  F(0,4);  #maximum value on the rectangle
```

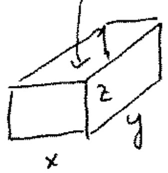
-38.50000000

-38.50000000

6.66666667

```
>
```

Example



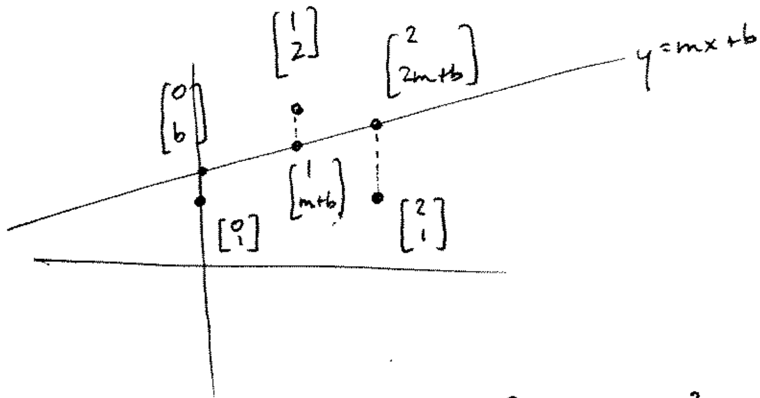

open-topped box.

Volume to be $4 \ m^3$

What dim's $x, y, z$ minimize surface area (cardboard area)?

## Example.

What line minimizes the "squared vertical deviations" for points $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$




$$F(m,b) = (b-1)^2 + (m+b-2)^2 + (2m+b-1)^2$$

Homework for Wed 10/20

§ 13.4 7, 9, 15, 22, 32, 38, 41, 43, 45–50, 55, 56, 57, 59

§ 13.5 9, 11, 14, 15, 25, 35, 37, 39, 52, 59, 61

§ 13.6 17, 24, 29, 36, 38

And.

I. page 991 #51 (linear regression in general).

(a) Show that $m$ and $b$ satisfy the matrix eqtn

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

(b) invert the 2x2 matrix above to find $m$ and $b$.

(c) verify that our class example (page 5) values are reproduced by this general formula.

II. Consider the lines

$$\vec{r}(t) = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{e}(s) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Find the nearest the lines approach each other by minimizing

$$|\vec{r}(t) - \vec{e}(s)|^2 .$$