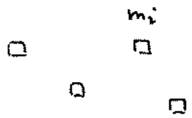
Math 2210-1
Monday 11/8
§14.5
moments, centers of mass, volume
Applications of double integrals:

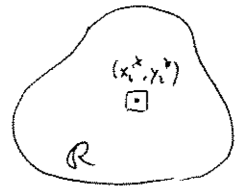
Mass

discrete



total mass $= \sum m_i$

lamina



$$m \approx \sum_i \underbrace{\delta(x_i^*, y_i^*)}_{\text{density}} \underbrace{\Delta A_i}_{\text{area}}$$

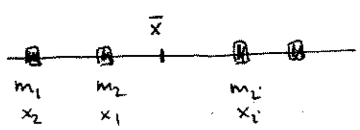
(mass/area)(area)

$$\boxed{m = \iint_R \delta(x,y)\, dA}$$

where $\delta(x,y)$ is the density function.

centers of mass

on a line:



$\bar{x}$ satisfies no net torque condition:

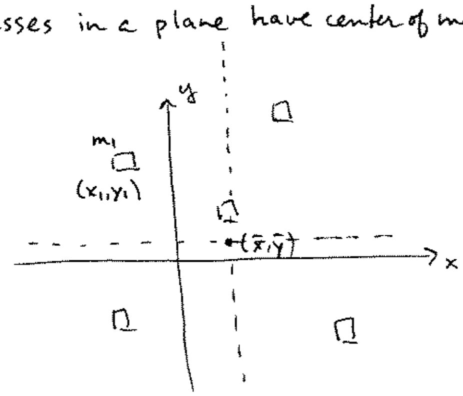
$$\sum_i m_i (x_i - \bar{x}) = 0$$

i.e.

$$\sum m_i x_i - \bar{x} \sum m_i = 0$$

$$\boxed{\bar{x} = \frac{\sum m_i x_i}{\sum m_i}}$$

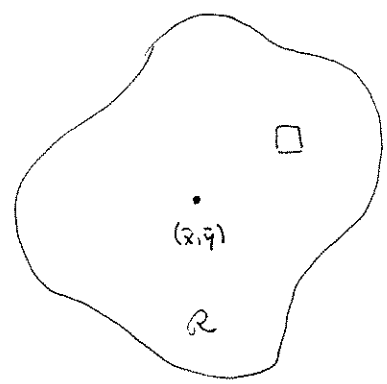
masses in a plane have center of mass $(\bar{x}, \bar{y})$



$$\sum m_i (x_i - \bar{x}) = 0$$
$$\sum m_i (y_i - \bar{y}) = 0$$

moment wrt y-axis

So, $$\boxed{\begin{aligned} \bar{x} &= \frac{\sum m_i x_i}{m} = \frac{M_{y\,axis}}{m} \\ \bar{y} &= \frac{\sum m_i y_i}{m} = \frac{M_{x\,axis}}{m} \end{aligned}}$$



$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$ defined by:

$$\iint_R (x - \bar{x})\, \delta(x,y)\, dA = 0$$

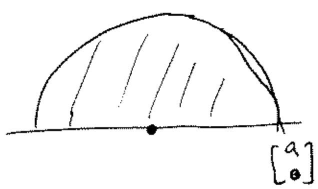
and $$\iint_R (y - \bar{y})\, \delta(x,y)\, dA = 0$$

yields

$$\boxed{\begin{aligned} \bar{x} &= \frac{\iint_R x\, \delta(x,y)\, dA}{m} \\ \bar{y} &= \frac{\iint_R y\, \delta(x,y)\, dA}{m} \end{aligned}}$$

<u>Remarks</u> if the density is constant in a lamina, then this constant can be factored out of the numerator & denom of the formulas for $\bar{x}$ & $\bar{y}$, so we may as well assume $\delta \equiv 1$ to compute $\bar{x}, \bar{y}$ in this case.

<u>Example</u> : Find the centroid of the uniform half disk



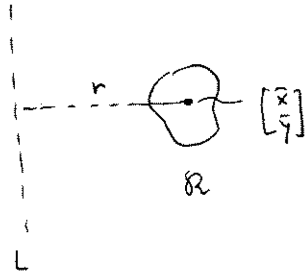
$$x^2+y^2 \le a^2$$
$$y \geq 0$$

- Explain why $\bar{x} = 0$

- Find $\bar{y}$

Volumes of revolution

Pappus' theorem I: Let $\mathcal{R}$ be a region in the plane, L be an axis on one side of this region



Consider the volume of revolution obtained by rotating $\mathcal{R}$ about L

Then volume

$$V = (2\pi r)\, A$$

where A is area of region

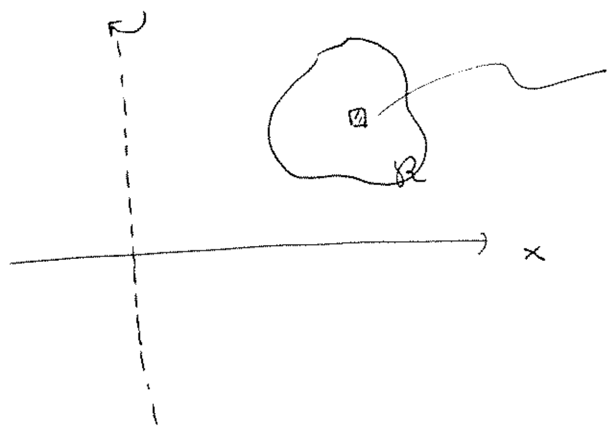
r = distance from axis to <u>centroid</u> $\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}$.

<u>example</u>: Use page 2 example to verify the centroid computation you did there is consistent with Pappus, since volume of radius a-ball is $\frac{4}{3}\pi a^3$.

<u>proof of Pappus I</u>

assume y-axis is axis of rotation.

and $x > 0$ for $\mathcal{R}$ points



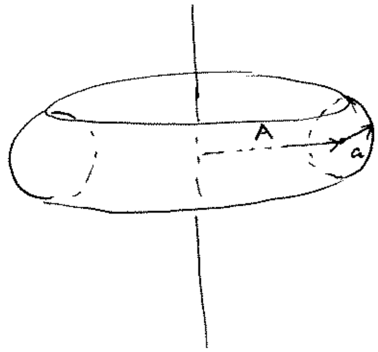
$$dV = (\underset{\substack{\downarrow\\ \text{circum.}}}{2\pi x})\, dA$$

$$V = \iint_{\mathcal{R}} 2\pi x \, dA$$

$$= 2\pi \underbrace{\iint_{\mathcal{R}} x \, dA}_{= \bar{x} A} !$$

$\blacksquare$

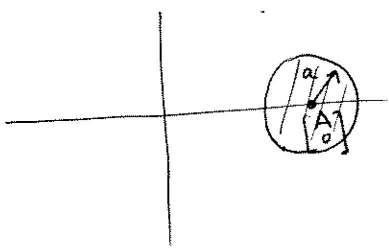
Example What is the volume of this torus (doughnut)?



obtained by rotating the disk

$$(x-A)^2+y^2 \le a^2$$

about the y-axis.



there are other applications in this section: moments of inertia, 2nd Pappus thm, which we won't cover in class (or hw or exams).