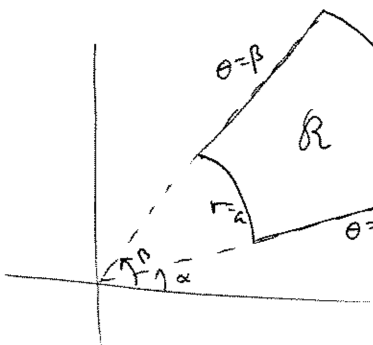
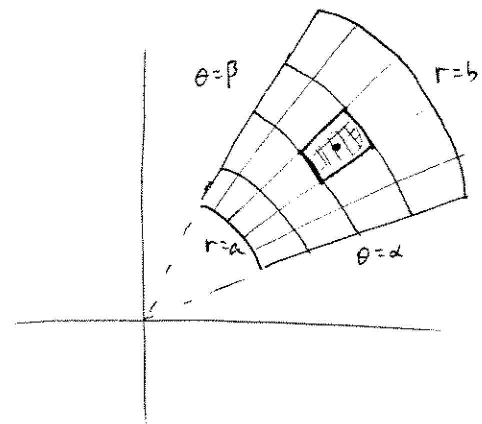
Math 2210-1
Friday 5 Nov
§14.4 Double integrals in polar coords

Consider a "polar rectangle"
$\alpha \le \theta \le \beta$
$a \le r \le b.$




partition this region using level curves of r and θ




We wish to compute

$$\boxed{\iint_R f(x,y)\,dA}$$

via a change of variables to polar coords

$x = r\cos\theta$
$y = r\sin\theta$

Riemann sum - sample at "midpoints"

$$\sum_{\substack{1\le i\le m \\ 1\le j\le n}} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^*\, \Delta r_i \Delta\theta_j$$

This is a Riemann sum for

$$g(r,\theta) = f(r\cos\theta, r\sin\theta)\underset{\text{extra term!}}{r}$$

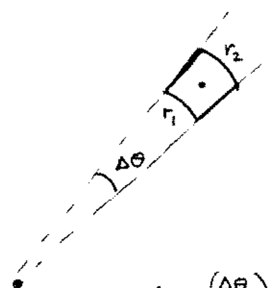
in a rectangle $\begin{array}{l} a \le r \le b \\ \alpha \le \theta \le \beta \end{array}$ of the r-θ plane.

As the partitioning size → 0, these Riemann sums converge to

$$\boxed{\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

$$= \iint_R f(x,y)\,dA$$

Exact computation of area:

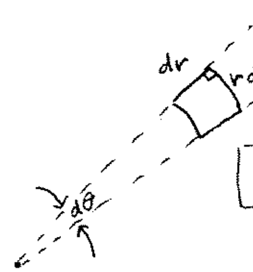



$$\Delta A = \underset{\substack{\uparrow \\ \text{fraction of} \\ \text{complete disk}}}{\left(\frac{\Delta\theta}{2\pi}\right)} \left(\underset{\substack{\text{radius } r_2\text{-disk area} \\ \downarrow}}{\pi r_2^2} - \underset{\substack{r_1 \text{ disk area} \\ \downarrow}}{\pi r_1^2}\right)$$

$$= \frac{\Delta\theta}{\cancel{1}} (r_2 - r_1)\frac{(r_2+r_1)}{2}$$

$$= (\Delta r)(\Delta\theta)\,\underset{\substack{\uparrow \\ \text{average radius}}}{\bar{r}}$$

Area picture with differentials (affine approx)



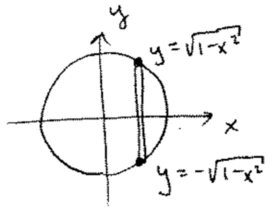

(an arc of radius r and angle dθ has length r dθ)

$$\boxed{dA = r\,dr\,d\theta}$$

Example : Verify that the area of the unit disk is $\pi$, using integration

(a)

old way




$$A = \iint_{\text{disk}} 1 \, dx\, dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \; dy \, dx = \int_{-1}^{1} 2\sqrt{1-x^2}\, dx$$

$$= 2\left[ \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x \right]_{-1}^{1}$$

(integral table # 54)

$$= \sin^{-1} 1 - \sin^{-1}(-1)$$

$$= \frac{\pi}{2} - -\frac{\pi}{2} = \pi \quad !!$$

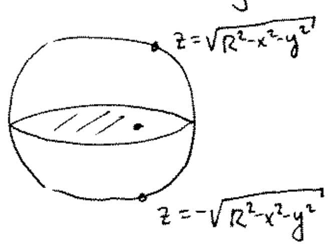
(b) polar coords C.O.V.

$$\iint_{\text{disk}} 1 \, dx\, dy = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta \qquad \tfrac{1}{2} r^2 \Big]_0^1$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta$$

$$= \pi \quad !$$

Example Verify that the volume of a radius R ball is $\frac{4}{3}\pi R^3$, using a double integral.




$$V = \iint_{x^2+y^2 \le R^2} 2\sqrt{R^2-x^2-y^2} \, dA$$

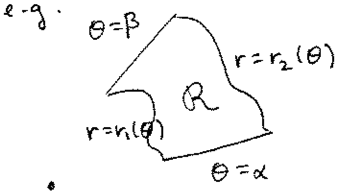
$x = r\cos\theta$

$y = r\sin\theta$

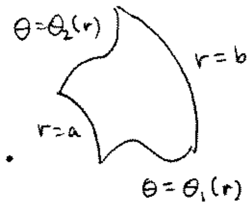
convert:

By the same reasoning as on page 1, you can change to polar coords if your region is radially or "angularly" simple

e.g.



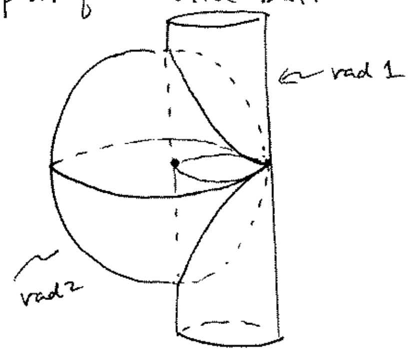

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

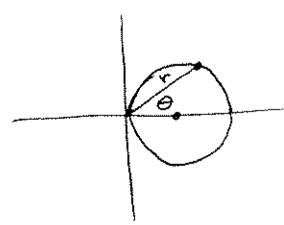



$$\iint_R f(x,y)\,dA = \int_a^b \int_{\theta_1(r)}^{\theta_2(r)} f(r\cos\theta, r\sin\theta)\, r\,d\theta\,dr.$$

example 3 p. 1018

an off center cylinder cores out part of a solid ball: Find its volume!



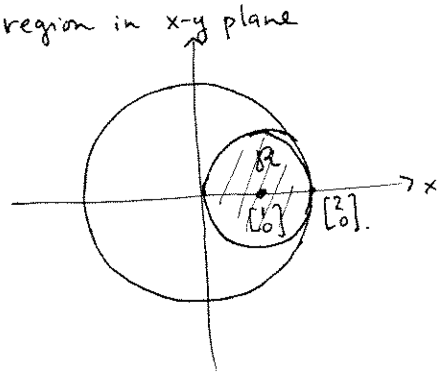

$(x-1)^2+y^2=1$

$x^2+y^2-2x=0$

$r^2-2r\cos\theta=0$

$$\boxed{\begin{array}{c} r=2\cos\theta \\ -\pi/2 \le \theta \le \pi/2 \end{array}}$$

region in x-y plane




$$\text{Vol} = \iint_R 2\sqrt{4-x^2-y^2}\,dA$$

$$\text{Vol} = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta}$$

... will need integral table 113