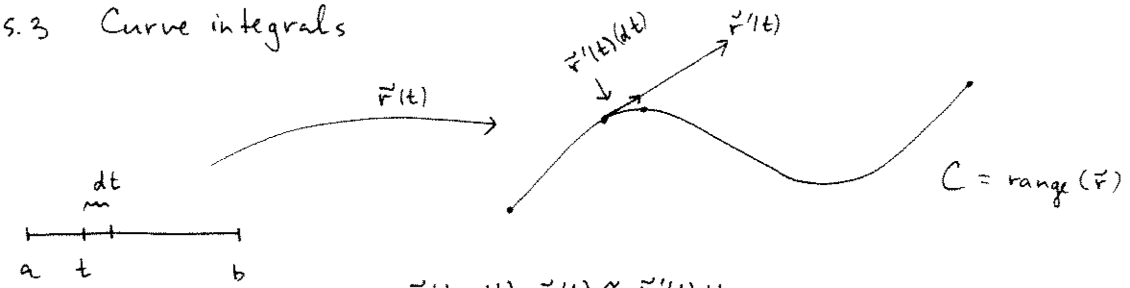
Math 2210-1
Monday 11/29.

§ 15.2–15.3 Curve integrals



$$\vec{r}(t+dt) - \vec{r}(t) \approx \vec{r}'(t)dt$$

$$\boxed{ds = |\vec{r}'(t)|\,dt}$$

Last Wednesday we did examples of curve integrals of type

$$\int_C f(\vec{x})\,ds \ := \int_a^b f(\vec{r}(t))\,|\vec{r}'(t)|\,dt$$

f a scalar fcn.
e.g. to find mass or moments of a wire.

Today: "Line integrals"

Let $\vec{F}(\vec{x})$ be a vector field defined near the curve C.

Then

$$\int_C \underbrace{\vec{F}\cdot \overset{\uparrow}{d\vec{x}}}_{} \ := \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

book writes $d\vec{r}$

If $\vec{F} = \langle P, Q, R\rangle$ this is also written as

$$\int_C P\,dx + Q\,dy + R\,dz$$

easy to remember if we just add $\boxed{d\vec{x} = \vec{r}'(t)dt}$ to our parametric picture at the top of the page.

<u>Physics connection</u>: $\int_C \vec{F}\cdot d\vec{x}$ is defined to be the work <u>W</u> done <u>by</u> the vector field $\vec{F}$ in moving a particle along C.
(This is defined to be the opposite of the work done by the particle.)

<u>Example 1</u> Consider the inverse square force field of attraction to the origin:

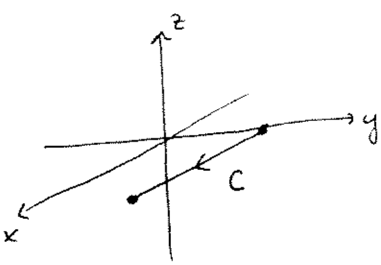
$$\vec{F}(x,y,z) = \frac{-c\,\vec{r}}{|\vec{r}|^3} = \frac{-c}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle. \qquad (c>0)$$

Let C be the line segment from $\begin{bmatrix}0\\3\\0\end{bmatrix}$ to $\begin{bmatrix}4\\3\\0\end{bmatrix}$.

Find the work done by the field in moving a particle along C.

(1) Take $\vec{r}(t) = \begin{bmatrix}0\\3\\0\end{bmatrix} + \begin{bmatrix}t\\0\\0\end{bmatrix}$, $0 \le t \le 4$

i.e. $x = t$
$y = 3$
$z = 0$



$$\vec{F}(\vec{r}(t)) = \frac{-c}{(t^2+9)^{3/2}}\langle t, 3, 0\rangle$$

$$\vec{r}\,'(t) = \begin{bmatrix}1\\0\\0\end{bmatrix} = \langle 1,0,0\rangle \qquad \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \frac{-ct}{(t^2+9)^{3/2}}$$

$$\int_C \vec{F}\cdot d\vec{x} = \int_0^4 -\frac{ct}{(t^2+9)^{3/2}}\,dt = c\,(t^2+9)^{-1/2}\frac{(+2)}{2}\Bigg]_0^4 = c\left[\frac{1}{5}-\frac{1}{3}\right] = \frac{-2c}{15}$$

notice work done by field is <u>negative</u>
(so work done by object is positive).

(2) "Same" computation.

use "x" as the parameter.

$$\vec{r}(x) = \begin{bmatrix}x\\3\\0\end{bmatrix} \qquad \int_C P\,dx + Q\,dy + R\,dz = \int_{x=0}^{4} \frac{-c}{(x^2+9)^{3/2}}\,x\,dx = \text{same answer.}$$

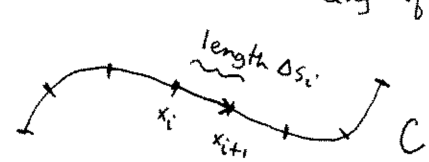
$dy = dz = 0.$

$\boxed{\text{Does it matter how you parameterize a curve?}}$

$\boxed{\text{So reparameterizations of the curve give same value for } \int_C f\,ds}$

$$\int_C f\,ds = \lim_{\|P\|\to 0} \sum_i f(\vec{x}_i)\,\Delta s_i$$

$\uparrow$ length of i^{th} segment

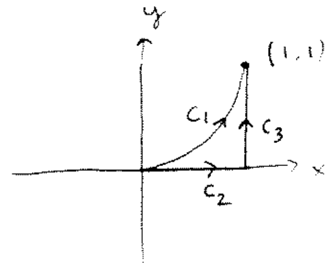


So, if you reverse direction you get the <u>opposite</u> value for this integral. $\int \vec{F}\cdot d\vec{x}$
direction preserving reparams yield same value

$$\int_C \vec{F}\cdot d\vec{x} = \lim_{\|P\|\to 0} \sum_i \vec{F}(\vec{x}_i)\cdot\underbrace{\Delta\vec{x}_i}_{\vec{x}_{i+1}-\vec{x}_i}$$

Example 2

$\vec{F}(x,y) = \langle 2x+y, -x \rangle.$



C_1 = parabola $y = x^2$ from $(0,0)$ to $(1,1)$

C_2 = line segment from $(0,0)$ to $(1,0)$
C_3 = ~~[illegible]~~ line segment from $(1,0)$ to $(1,1)$.

compute $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{x} = \int_{C_1} (2x+y)\,dx - x\,dy$

(ans = 2/3)

compute $\displaystyle\int_{C_2 \cup C_3} \vec{F} \cdot d\vec{x}$

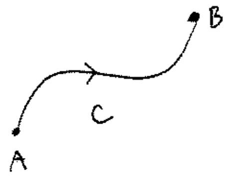
↑ book might write $C_2 + C_3$

(ans = 0).

Line integrals $\int_C \vec{F}\cdot d\vec{x}$ are called <u>path independent</u>

if the values <u>only</u> depend on the starting and final point of the curve, <u>not</u> the route taken.

<u>Theorem</u> If $\vec{F}$ is a gradient field, $\vec{F} = \nabla f$, then



$$\int_C \vec{F}\cdot d\vec{x} = f(B) - f(A)$$ where A is starting point & B is final point.

[In fact, every vector field which gives rise to path independent line integrals is a gradient field!].

<u>proof</u>. Let $\vec{r}: [a,b] \to \mathbb{R}^n$

parameterize C, $\vec{r}(a) = A$, $\vec{r}(b) = B$, & $\vec{F} = \nabla f$

$$\int_C \vec{F}\cdot d\vec{x} := \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \underbrace{\nabla f(\vec{r}(t))\cdot \vec{r}\,'(t)}_{\frac{d}{dt} f(\vec{r}(t)) \text{ by multivar. chain rule!}}\,dt$$

$$= f(\vec{r}(t))\Big]_a^b = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(B) - f(A).$$

<u>Example 1 page 2</u>

<u>examples</u>: $\vec{F} = -\dfrac{c\vec{r}}{|\vec{r}|^3}$ <u>is</u> a gradient field,

$$\vec{F} = \nabla\left(+\frac{c}{|\vec{r}|}\right) = \nabla\Big(\underbrace{\frac{c}{\sqrt{x^2+y^2+z^2}}}_{\text{"}f\text{"}}\Big).$$

so, $$\int_{\left[\begin{smallmatrix}0\\3\\0\end{smallmatrix}\right]}^{\left[\begin{smallmatrix}4\\3\\0\end{smallmatrix}\right]} \vec{F}\cdot d\vec{x}$$

$$= f(B) - f(A)$$

$$= c\left(\frac{1}{5} - \frac{1}{3}\right) \checkmark.$$

<u>Example 2</u> cannot be a gradient field, by our computations on page 3.

Also, $\left.\begin{aligned} f_x &= (2x+y) \\ f_y &= -x \end{aligned}\right\}$ not possible! (?).