Math 2210-1
Wed Nov. 24
15.1 - 15.2 (Monday after Thanksgiving will continue 15.2 & start 15.3)

HW for Wed 12/1
15.1 1, 5, 6, 11-14, 15, 17, 22, 25-30, 35, 37, 38
15.2 1, 3, 7, 11, 15, 19, 21

Vector fields

$\vec{F}(x,y) = \langle P, Q \rangle$ $\quad \mathbb{R}^2$
$\vec{F}(x,y,z) = \langle P, Q, R \rangle$ $\quad \mathbb{R}^3$

$\vec{\nabla}$ "del"
" $\langle \partial_x, \partial_y \rangle$ $\quad \mathbb{R}^2$
$\langle \partial_x, \partial_y, \partial_z \rangle$ $\quad \mathbb{R}^3$

$$\left.\begin{array}{ll} f : \mathbb{R}^2 \to \mathbb{R}, \quad \vec{\nabla} f = \langle f_x, f_y \rangle & (\text{think } \langle \partial_x, \partial_y \rangle f = \langle \partial_x f, \partial_y f \rangle) \\ f : \mathbb{R}^3 \to \mathbb{R} \quad \vec{\nabla} f = \langle f_x, f_y, f_z \rangle & (\text{think } \langle \partial_x, \partial_y, \partial_z \rangle f = \langle \partial_x f, \partial_y f, \partial_z f \rangle \end{array}\right\} \text{gradient of a function}$$

$$\left.\begin{array}{l} \vec{F} = \langle P, Q \rangle \\ \quad \vec{\nabla} \cdot \vec{F} = \langle \partial_x, \partial_y \rangle \cdot \langle P, Q \rangle = P_x + Q_y = \operatorname{div} \vec{F} \\ \vec{F} = \langle P, Q, R \rangle \\ \quad \vec{\nabla} \cdot \vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z = \operatorname{div} \vec{F} \end{array}\right\} \text{divergence of a vector field.}$$

$$\left.\begin{array}{l} \vec{F} = \langle P, Q, R \rangle \\ \quad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ \quad n=2, \ \vec{F} = \langle P, Q \rangle, \ \text{scalar curl} \vec{F} := Q_x - P_y \\ \qquad \left( \text{if we extended to } \mathbb{R}^3, \ \vec{F} = \langle P(x,y), Q(x,y), 0 \rangle \atop \text{then } \vec{\nabla} \times \vec{F} = \langle 0, 0, Q_x - P_y \rangle \right) \end{array}\right\} \text{curl of a vector field.}$$

Check If $\vec{F} = \nabla f$ is a gradient vector field (also called a conservative vector field) (and then $f$ is called the potential function)
then $\vec{\nabla} \times \vec{F} = \vec{0}$. That is, $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$

So, curl $\vec{F}$ measures whether $\vec{F}$ could be a gradient vector field.
It also measures "rotation" in a vector field,

i.e. suppose the vector field is a field of tangent vectors for moving particles, as in a fluid flow.
It turns out curl $\vec{F}$ is a vector which measures how much they're rotating.

e.g. $\vec{F} = \langle -y, x, 0 \rangle$

$\vec{\nabla} \times \vec{F} =$

like a 3-d version of picture 2 page 4 Wed.

div $\vec{F}$ measures whether your fluid (e.g. a gas) is expanding or contracting, e.g.

$\vec{\nabla} \cdot \langle x, y \rangle = 2$

picture 1 page 4 Wed.

<u>Example</u> in $\mathbb{R}^3$, $f(x,y,z) = \dfrac{-1}{\sqrt{x^2+y^2+z^2}}$ $\quad \left(= -\frac{1}{e}\right)$.

Compute $\nabla f = \vec{F}$

notice this is the inverse square law (of gravity & ~~electrostatics~~ electric charge).

Thus $\vec{\nabla} \times \vec{F} = \vec{0}$.
Show also $\vec{\nabla} \cdot \vec{F} = 0$ (in HW).

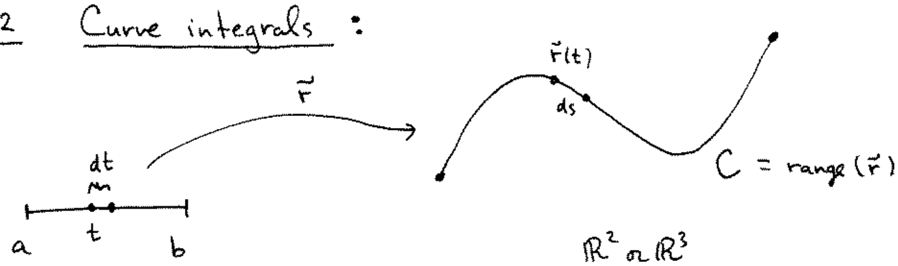
**Example**

$$\vec{F} = \langle 2x+y,\ e^x,\ 2z \rangle$$

Compute

$$\vec{\nabla} \times \vec{F}$$
$$\vec{\nabla} \cdot \vec{F}$$

## §15.2 Curve integrals:




recall the element of arclength on $C$,

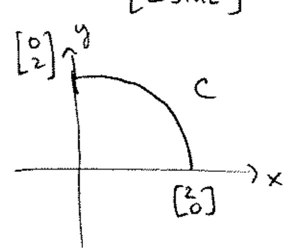
$$\boxed{ds = |\vec{r}\,'(t)|\,dt}$$

so if $f: C \to \mathbb{R}$ is a <u>scalar</u> function,

$$\boxed{\int_C f\, ds \ := \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$$

**Example 1** $C$ is a radius 2 quarter circle, $\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix}$, $0 \le t \le \pi/2$

Compute $\displaystyle\int_C \frac{xy}{2}\, ds$



<u>Example 2</u> suppose that $\frac{xy}{2}$ was the density of the (wire) curve in example 1

e.g. grams/meter

$\delta$ = mass/length since curve is 1 dim'l.

then we computed the wire's mass in Example 1.

Set up integrals for the wire's centroid
and its moment of intertia about the y-axis.

$$m = \int_C \delta \, ds$$

$$\bar{x} = \frac{\int_C \delta x \, ds}{m}$$

$$\bar{y} = \frac{1}{m} \int_C \delta y \, ds$$

$$\bar{z} =$$

$$I_y = \int_C \delta x^2 \, ds$$

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> mass:=int(4*cos(t)*sin(t),t=0..Pi/2);
  xbar:=int(8*cos(t)^2*sin(t),t=0..Pi/2)/mass;
  ybar:=int(8*cos(t)*sin(t)^2,t=0..Pi/2)/mass;
  Iy:=int(16*cos(t)^2*sin(t)*cos(t),t=0..Pi/2);
>
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$$mass := 2$$

$$xbar := \frac{4}{3}$$

$$ybar := \frac{4}{3}$$

$$Iy := 4$$

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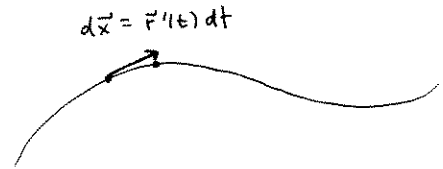
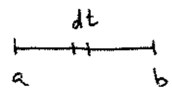
The second kind of curve integral arises in computations of <u>work</u>.

$$\int_C \vec{F}\cdot d\vec{x} = \begin{cases} \int_C P\,dx + Q\,dy & n=2 \\ \int_C P\,dx + Q\,dy + R\,dz & n=3 \end{cases}$$




$$\vec{x} = \vec{r}(t)$$

$$\Delta\vec{x} \approx d\vec{x} = \vec{r}\,'(t)\,dt$$




$$\boxed{\int_C \vec{F}\cdot d\vec{x} := \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}$$

So this can be thought of as

$$\int_a^b \underbrace{\vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}_{\text{"}f\text{"}} \;\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

$$= \int_C (\vec{F}\cdot\vec{T})\,ds$$

unit tang. vector.

We discuss these on Monday