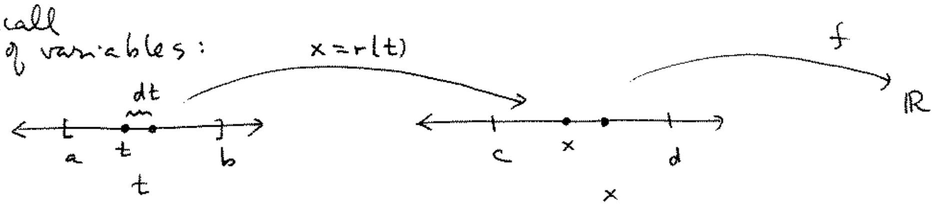
Math 2210-1
Monday 11/22
14.9, 15.1.

↓ Recall
change of variables:

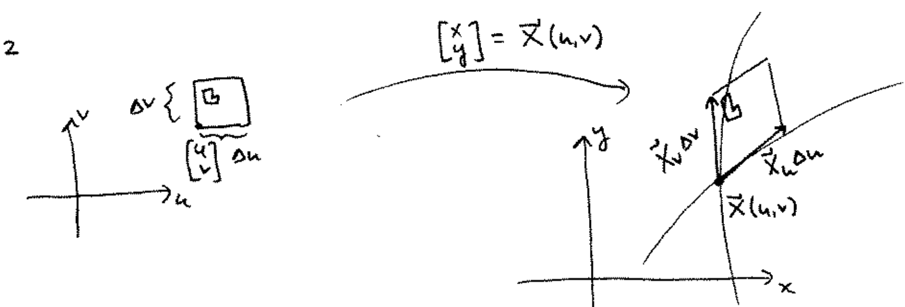
$n=1$



$x = r(t)$, r increasing
$\boxed{dx = r'(t)\,dt}$

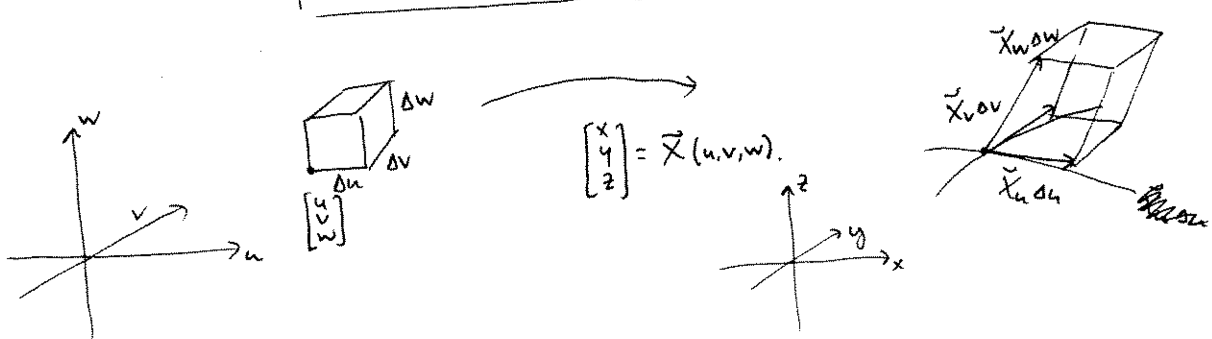
$$\int_c^d f(x)\,dx = \int_a^b f(r(t))\,r'(t)\,dt$$

$n=2$



$$\boxed{\begin{aligned} dA &= \left|\det X'(u,v)\right|\,du\,dv \\ &= \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv \end{aligned}}$$

$n=3$



$$\boxed{\begin{aligned} dV &= \left|\det\left(X'(u,v,w)\right)\right|\,du\,dv\,dw \\ &= \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,du\,dv\,dw \end{aligned}}$$

Examples

1. Rederive $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ for spherical coords.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ \rho \cos\phi \end{bmatrix}$$

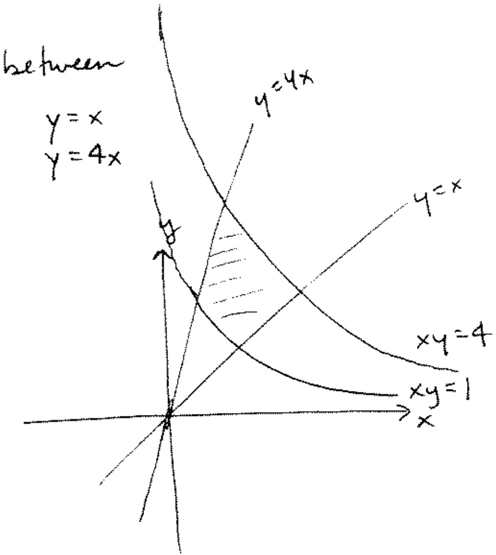
2. Rederive $dV = r \, dr \, d\theta \, dz$ for cylindrical coords

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}.$$

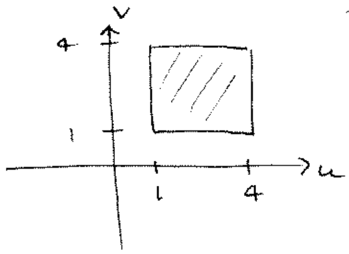
Example 3

Find the area in the curved quadrilateral between

$$xy=1 \qquad y=x$$
$$xy=4 \qquad y=4x$$



let $u=xy$
$v=y/x$



?

$$\text{Area} = \int_1^4\int_1^4 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

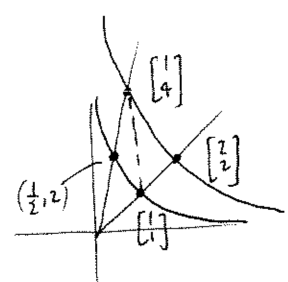
$$u = xy \qquad uv = y^2 \rightarrow \boxed{y = (uv)^{1/2}}$$
$$v = y/x \qquad u/v = x^2 \rightarrow \boxed{x = u^{1/2}v^{-1/2}}$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix}$$

$$= \frac{1}{4}v^{-1} + \frac{1}{4}v^{-1} = \frac{1}{2}v^{-1}$$

$$= \int_1^4\int_1^4 \frac{1}{2}v^{-1}\,du\,dv$$
$$= 3\int_1^4 \frac{1}{2}v^{-1}dv = \frac{3}{2}\ln v\Big]_1^4 = \boxed{3\ln 2}$$

check:

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> int(4*x-1/x,x=1/2..1)+int(4/x-x,x=1..2);
                                              3 ln(2)
>
```

Note: sometimes it is easy to see (u,v) in terms of (x,y), but solving for the inverse fcn is "impossible". Luckily, the chain rule says that the derivative matrices of inverse fcns are inverse matrices. Also, the det. of matrix products is the product of the dets. In our case

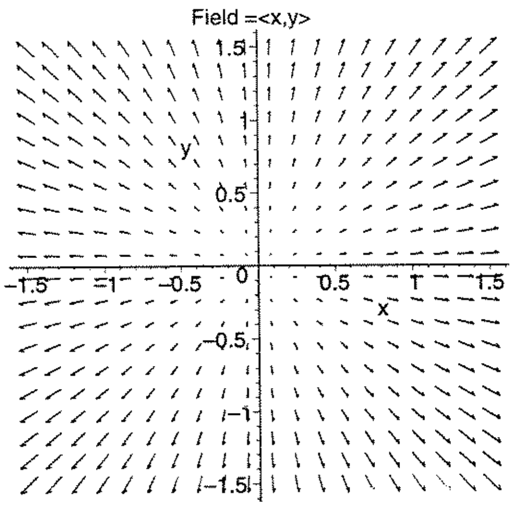
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x} = 2v$$

so we deduce $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2v}$!

§15.1 Vector fields.

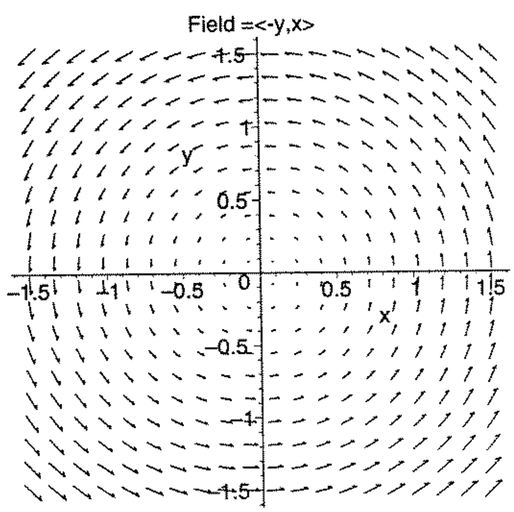
$\vec{F}(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z) \rangle$ vector fields in $\mathbb{R}^3$

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ v. fields in $\mathbb{R}^2$.



$\vec{F} = \langle P, Q \rangle = \langle x, y \rangle$

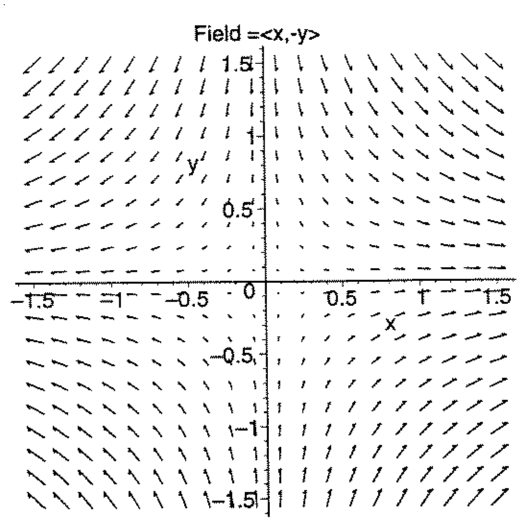
gradient field? $(\vec{F} = \nabla f)$



$\langle P, Q \rangle = \langle -y, x \rangle$.

gradient field?

(Think Escher!)



$\langle P, Q \rangle = \langle x, -y \rangle$

gradient field?

Think of ∇ meaning $\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle = \langle \partial_x, \partial_y, \partial_z \rangle$ in $\mathbb{R}^3$

$\boxed{\text{divergence}}$ or $\langle \partial_x, \partial_y \rangle$ in $\mathbb{R}^2$

$\downarrow$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad \mathbb{R}^3$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \qquad \mathbb{R}^2$$

$\boxed{\text{curl}}$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \quad \text{in } \mathbb{R}^3$$

in $\mathbb{R}^2$, scalar $\underset{\operatorname{curl}\vec{F}}{\text{curl}}$ is $Q_x - P_y$.

<u>note</u> if $\vec{F} = \nabla f = \langle f_x, f_y, f_z \rangle$, then $\operatorname{curl} \vec{F} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$

$$= \vec{0}\,!$$

Compute div, curl for our 3 $\mathbb{R}^2$ vector fields. Think about what they may mean. (in relation to the English meaning of <u>divergence</u> and <u>curl</u>)