Math 2210-1
Wed Nov. 17

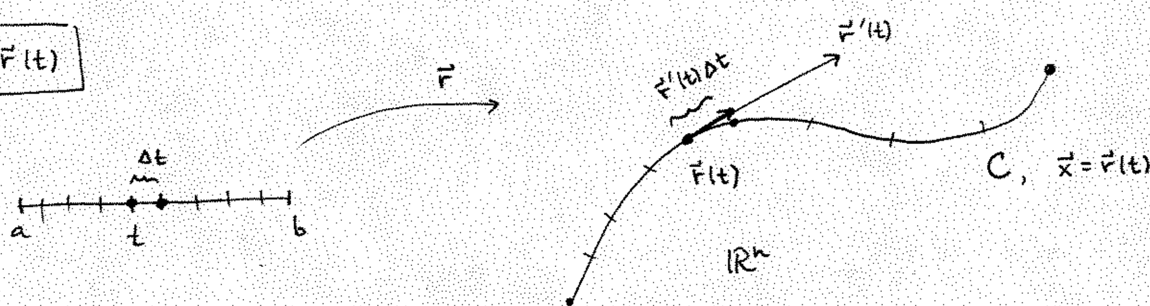
HW for Wed 11/24
§14.9 (3, 7, 8, 13, 14, 22)

Practice exam review session
tomorrow (Thurs.)
LCB 121, 9:40-10:30

§14.9 Change of variables in multiple integration

this discussion fits into a bigger picture discussion of integrals for parametric objects:

- curves $\vec{r}(t)$



$\vec{r}(t+\Delta t) \approx \vec{r}(t) + \vec{r}\,'(t)\Delta t$ (ignoring "error")

$\Delta\vec{r} \approx \vec{r}\,'(t)\Delta t$

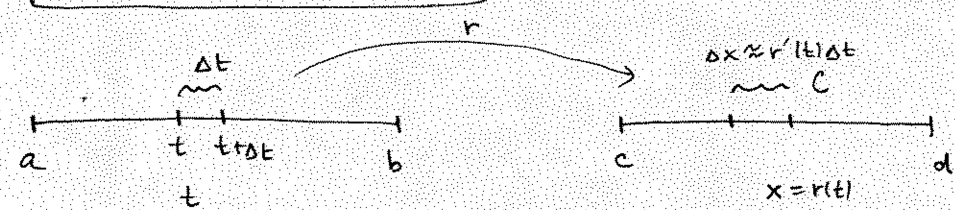
$ds = |\vec{r}\,'(t)|\,dt$ element of arclength

(Section 12.2) length $\boxed{L = \int_a^b |\vec{r}\,'(t)|\,dt}$

(15.2) More generally, $\boxed{\int_C f(\vec{x})\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$

e.g. f could be scalar density fcn (mass/length) and you could be computing total mass.

Special case is change of variables in 1-var. Calc:

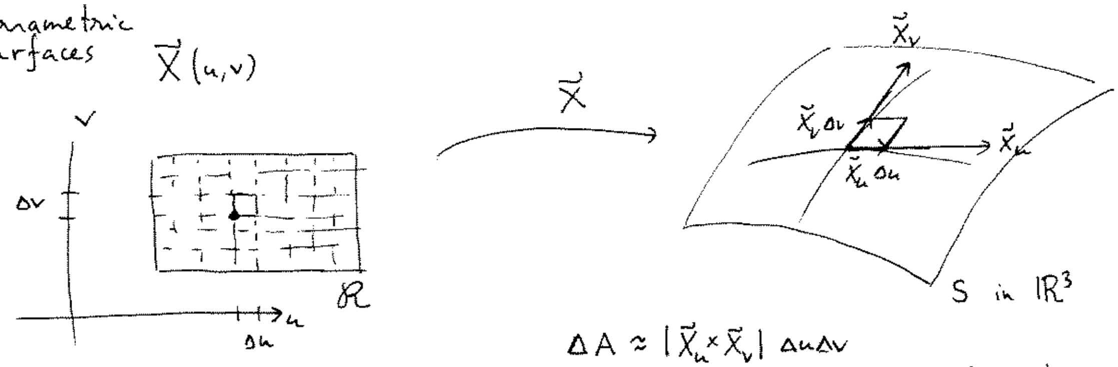


picture when $\underline{r'(t)>0}$ on $[a,b]$.

$$\int_C f(x)\,ds = \int_c^d f(x)\,dx = \int_a^b f(r(t))\,r'(t)\,dt$$

i.e. you've substituted
$x = r(t)$
$dx = r'(t)\,dt$

- parametric surfaces $\vec{X}(u,v)$



$\Delta A \approx |\vec{X}_u \times \vec{X}_v| \, \Delta u \Delta v$

$dA = |\vec{X}_u \times \vec{X}_v| \, du dv$ area element

(based on $\vec{X}(u+\Delta u, v+\Delta v) \approx \vec{X}(u,v) + \underbrace{\left[\vec{X}_u \middle| \vec{X}_v \right] \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}}_{= \vec{X}_u \Delta u + \vec{X}_v \Delta v}$)

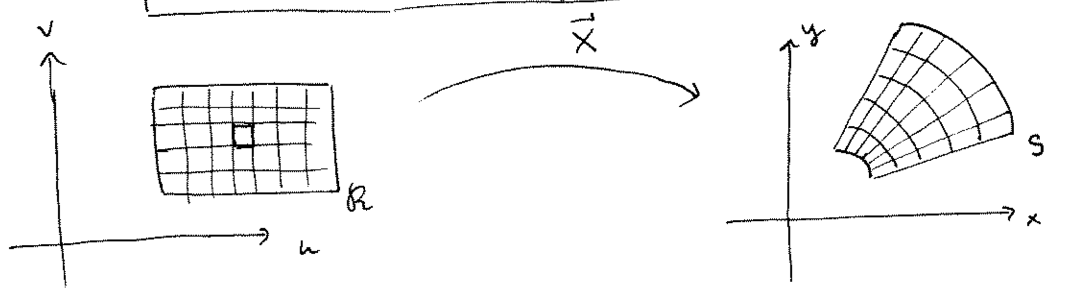
so,

§ 14.8 $\boxed{A = \iint\limits_{R} |\vec{X}_u \times \vec{X}_v| \, du dv}$

more generally,

§ 15.5 $\boxed{\iint\limits_{S} f(\vec{x}) \, dA = \iint\limits_{R} f(\vec{X}(u,v)) \, |\vec{X}_u \times \vec{X}_v| \, du dv}$

Special case is change of variables in 2-var integrals:



think of $\vec{X}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ 0 \end{bmatrix}$

$|\vec{X}_u \times \vec{X}_v| = \text{abs} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix}$

$= \text{abs}\left(\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right)$

$= \text{abs}\left(\underbrace{\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}}_{} \right)$

determinant of deriv matrix for $\begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$

called <u>Jacobian</u> det,

write $\dfrac{\partial(x,y)}{\partial(u,v)}$

so, $\boxed{\iint\limits_{S} f(x,y) \, dA = \iint\limits_{R} f(\vec{X}(u,v)) \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| du dv}$

Examples

(1) $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{X}\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

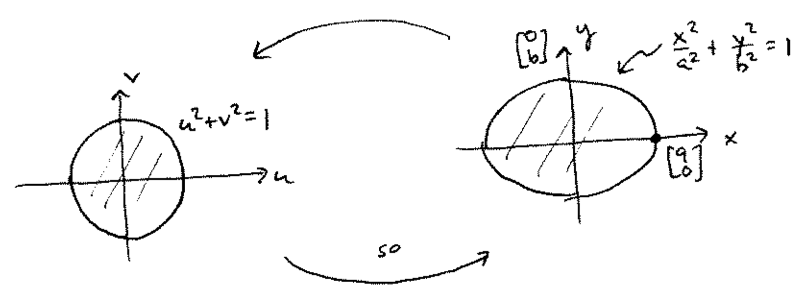
derivative matrix is :

determinant is :

deduce. $\displaystyle\iint\limits_{\substack{x-y \\ \text{region}}} f(x,y)\,dA = \iint\limits_{\substack{r-\theta \\ \text{region}}} f(r\cos\theta, r\sin\theta)\ \overset{\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|}{\overset{\downarrow}{r}}\ dr\,d\theta$

(2) Find the area inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$u = \frac{x}{a}, \ v = \frac{y}{b}$

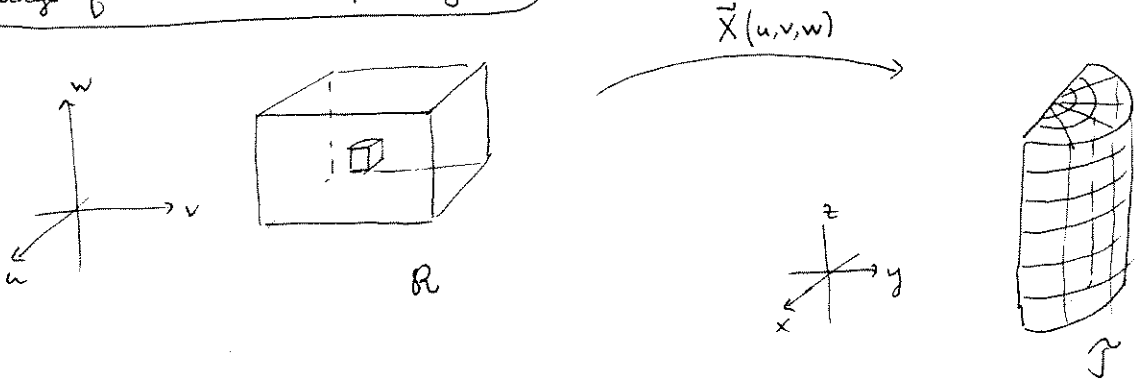


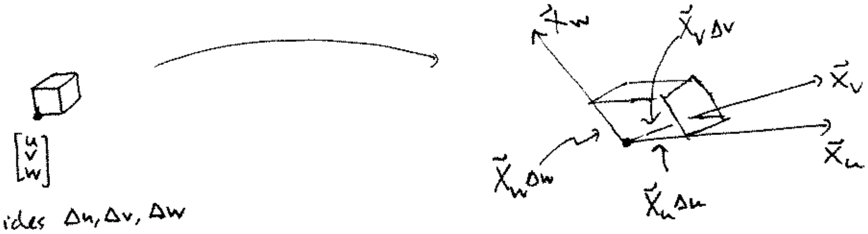
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} au \\ bv \end{bmatrix}$

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$; det = ab

so $\iint dA = \iint ab\, du\,dv = \boxed{ab\,\pi}$

Change of variables in triple integrals





sides $\Delta u, \Delta v, \Delta w$

$$\Delta V \approx \text{abs}\Big(\underbrace{\left|\vec{X}_u \,\vdots\, \vec{X}_v \,\vdots\, \vec{X}_w\right|}_{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|.}\Big)\,\Delta u\,\Delta v\,\Delta w$$

based on $\vec{X}(u+\Delta u, v+\Delta v, w+\Delta w) \approx \vec{X}(u,v,w) + \left[\vec{X}_u \,\vdots\, \vec{X}_v \,\vdots\, \vec{X}_w\right]\begin{bmatrix}\Delta u\\ \Delta v\\ \Delta w\end{bmatrix}.$

$$\iiint_{\mathcal{T}} f\,dV = \iiint_{\mathcal{R}} f(\vec{X}(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

<u>check</u>

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{X}\begin{bmatrix}r\\ \theta\\ z\end{bmatrix} = \begin{bmatrix}r\cos\theta\\ r\sin\theta\\ z\end{bmatrix} \qquad \text{deriv mat } \vec{X}' = \begin{bmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

det $= r$ ✓

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{X}\begin{bmatrix}\rho\\ \phi\\ \theta\end{bmatrix} = \begin{bmatrix}\rho\sin\phi\cos\theta\\ \rho\sin\phi\sin\theta\\ \rho\cos\phi\end{bmatrix} \qquad \left|\det(X')\right| = \rho^2\sin\phi \;\;!$$

Exam Review (Exam is Friday, in class. Closed book & note. No graphing calculators allowed. I'll provide you with the book's integral tables – you should know basic integrals & method of substitution.)

Double & Triple integrals

- Riemann sums → integrals are limits of these, so understanding of what integrals mean (area, volume, mass, moments, etc.) is based on finite Riemann sums.

- Computing iterated integrals (by partial integration) (relatively "easy")

- Converting a description of a region into an "iteration", so that you can write down iterated integrals for functions defined on that region (and vise-verse!) (harder!)

- change of variables to polar, cylindrical, or spherical coords

Know: $dA = r\,dr\,d\theta$, $dV = r\,dr\,d\theta\,dz$, $dV = \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$

- applications
  - mass (2-d, 3-d regions); includes area & volume
  - center of mass (for 2-d, 3-d regions) (centroid)
  - moments of inertia (3-d regions)
  - Surface area (for surfaces in space)

Know all formulas

$A = \iint_R 1\,dA$, $V = \iiint_R 1\,dV$

$m = \iint_R \delta\,dA$ or $\iiint_R \delta\,dV$

$\bar{x} = \frac{1}{m}\iint x\,\delta A$ (or $\frac{1}{m}\iiint_R x\,dV$)

$\bar{y} =$

$\bar{z} =$

$I_z = \iiint_R (x^2+y^2)\,dV$

$A = \iint_{u\text{-}v \text{ region}} |\vec{X}_u \times \vec{X}_v|\,du\,dv$

for $\vec{X}\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}x\\y\\f(x,y)\end{bmatrix}$

becomes

$A = \iint_{x\text{-}y \text{ region}} \sqrt{1+f_x^2+f_y^2}\,dx\,dy$

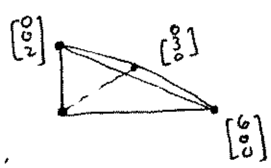
Sample test

Not inclusive!

1 a) Compute $\iint_R f\,dA$ if $f(x,y) = xy$ and R is the rectangle $1 \le x \le 2$, $0 \le y \le 2$

b) Change limits (by first sketching region), and then evaluate $\int_0^4 \int_{\sqrt{y}}^2 \frac{y e^{(x^2)}}{x^3}\,dx\,dy$

2. Consider the pyramid in the first octant bounded by the three coord planes and the skew plane $x+2y+3z=6$



a) Find its volume with a triple integral

b) Find the x-coord of its centroid (assume density $\equiv 1$).

3. Let D be the disk of radius 2 centered at the origin. Find $\iint_D \frac{1}{1+x^2+y^2}\,dA$