Math 2210-1

Monday Nov. 15

§14.8 : parametric surfaces and surface area

$$\vec{X} : R \longrightarrow \mathbb{R}^3, \quad R \subset \mathbb{R}^2$$

$$\vec{X}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

<u>Examples</u> :

1. $\vec{X}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$ graph of a function f i.e. points s.t. $z = f(x,y)$

   or $\vec{X}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$

2. $\vec{X}\begin{pmatrix} \theta \\ z \end{pmatrix} = \begin{bmatrix} a\cos\theta \\ a\sin\theta \\ z \end{bmatrix}$, $a > 0$ cylinder of radius a

3. $\vec{X}\begin{pmatrix} \phi \\ \theta \end{pmatrix} = \begin{bmatrix} a\sin\phi\cos\theta \\ a\sin\phi\sin\theta \\ a\cos\phi \end{bmatrix}$, $a > 0$ sphere of radius a

4. $\vec{X}\begin{pmatrix} t \\ \theta \end{pmatrix} = \begin{bmatrix} x(t)\cos\theta \\ x(t)\sin\theta \\ z(t) \end{bmatrix}$, $x(t) > 0$, surface of revolution abt z-axis.

5. $\vec{X}\begin{pmatrix} u \\ v \end{pmatrix} = u\vec{a} + v\vec{b} + \vec{c}$ plane thru $\vec{c}$ with dir. vectors $\vec{a}$ & $\vec{b}$.

Affine approximation geometry:

For $\vec{X}\begin{pmatrix}u\\v\end{pmatrix} = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$

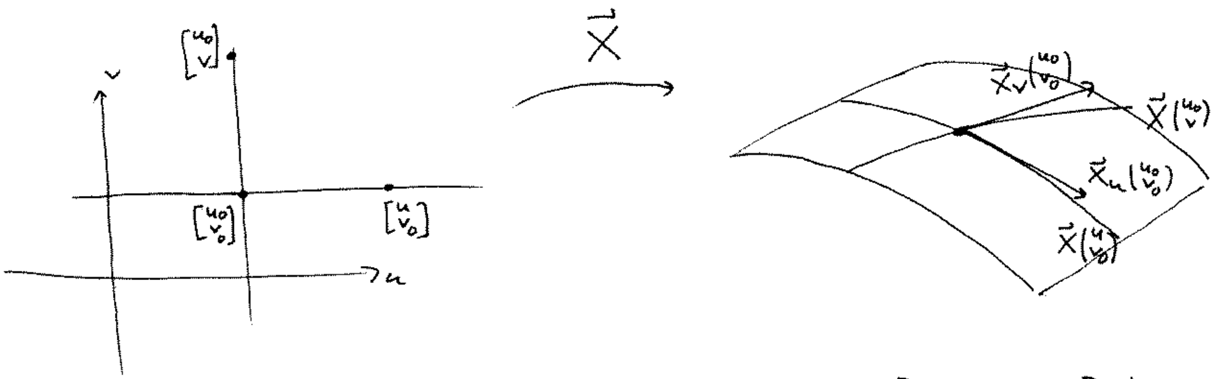
There are 2 natural tangent vectors to the range surface, namely

$$\frac{\partial}{\partial u}\vec{X} = \vec{X}_u := \lim_{\Delta u \to 0} \frac{\vec{X}(u+\Delta u, v) - \vec{X}(u,v)}{\Delta u} = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}$$

$$\frac{\partial}{\partial v}\vec{X} = \vec{X}_v := \lim_{\Delta v \to 0} \frac{\vec{X}(u, v+\Delta v) - \vec{X}(u,v)}{\Delta v} = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}$$

$\vec{X}_u\begin{pmatrix}u_0\\v_0\end{pmatrix}$ is the tangent to the curve $\vec{r}(u) = \vec{X}\begin{pmatrix}u\\v_0\end{pmatrix}$

$\vec{X}_v\begin{pmatrix}u_0\\v_0\end{pmatrix}$ is the tangent to the curve $\vec{\rho}(v) = \vec{X}\begin{pmatrix}u_0\\v\end{pmatrix}$



Domain

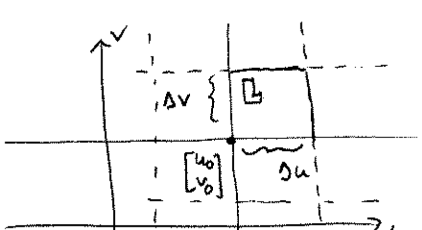
Range. : Deduce

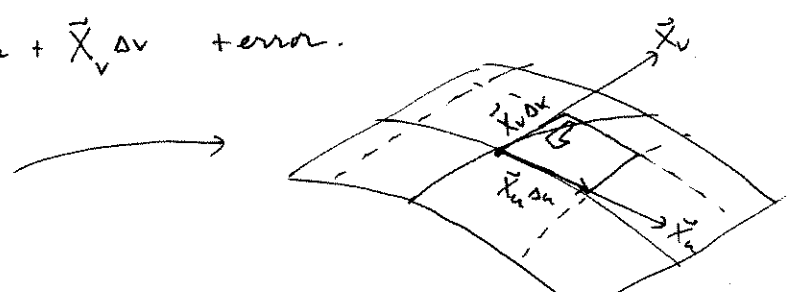
$\boxed{\vec{X}_u \times \vec{X}_v \text{ is normal vector to parametric surface}}$

Affine approx formula:

$$\vec{X}\begin{pmatrix}u+\Delta u\\v+\Delta v\end{pmatrix} = \vec{X}\begin{pmatrix}u\\v\end{pmatrix} + \begin{array}{cc} \vec{X}_u & \vec{X}_v \\ \downarrow & \downarrow \end{array} \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta v\end{bmatrix} + \text{error}$$

$$= \vec{X}\begin{pmatrix}u\\v\end{pmatrix} + \vec{X}_u \Delta u + \vec{X}_v \Delta v + \text{error}.$$





Deduce:

$$\Delta A \approx |\vec{X}_u \Delta u \times \vec{X}_v \Delta v|$$

$$= |\vec{X}_u \times \vec{X}_v|(\Delta u)(\Delta v).$$

could integrate other functions, e.g. $\delta\begin{pmatrix}x\\y\\z\end{pmatrix}$ density

so

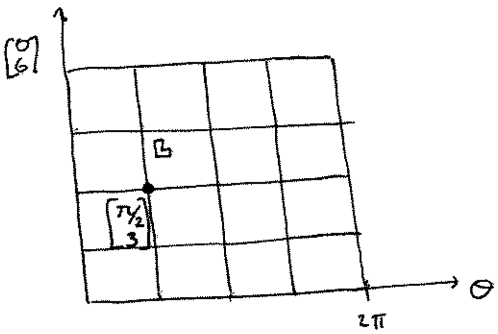
$$\boxed{\text{Area of parametric surface} = \iint_{\substack{u\text{-}v \text{ region} \\ \text{domain}}} 1 \cdot |X_u \times X_v| \, du\, dv}$$

Examples (from ②)

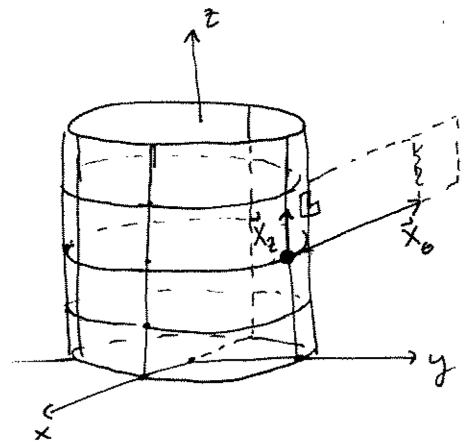
$$\vec{X}\begin{pmatrix}\theta \\ z\end{pmatrix} = \begin{bmatrix} 3\cos\theta \\ 3\sin\theta \\ z \end{bmatrix}$$

$0 \le \theta \le 2\pi$

$0 \le z \le 6$



Domain



$$\vec{X}_\theta = \begin{bmatrix} -3\sin\theta \\ 3\cos\theta \\ 0 \end{bmatrix}$$

$$\vec{X}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{X}\begin{pmatrix}\pi/2 \\ 3\end{pmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$$\vec{X}_\theta\begin{pmatrix}\pi/2 \\ 3\end{pmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{X}_z\begin{pmatrix}\pi/2 \\ 3\end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{X}_\theta \times \vec{X}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 3\cos\theta, 3\sin\theta, 0\rangle$$

$$\langle -3,0,0\rangle \times \langle 0,0,1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 0,3,0\rangle$$

normal vector at $\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$.

$$|\vec{X}_\theta \times \vec{X}_z| = 3$$

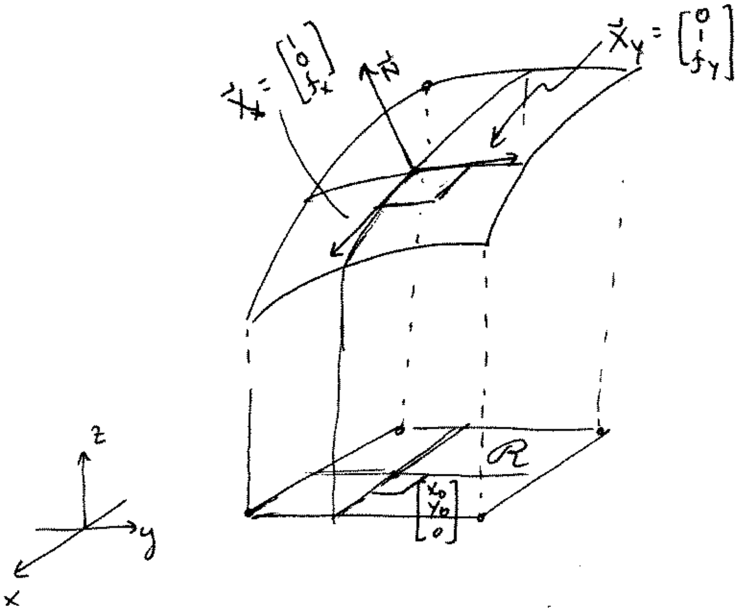
$$\text{image area} = \underbrace{\int_0^{2\pi} \underbrace{\int_0^6 1 \cdot \underbrace{|\vec{X}_\theta \times \vec{X}_z|}_{3}\, dz}_{18}\, d\theta}_{36\pi}$$

(is correct: $h \cdot \text{circum} = 6 \cdot 2\pi \cdot 3$)

Example

graph $z = f(x,y)$

$$\vec{X}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$$



$$\vec{X}_x \times \vec{X}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$ normal vector.

yet another way to get tangent plane at $(x_0, y_0, f(x_0,y_0))$:

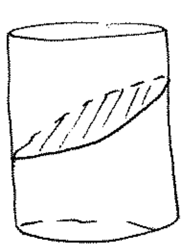
$$-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + (z-z_0) = 0$$

same as old eqtn after we rewrite it.

$$dA = |X_x \times X_y|\, dx\,dy = \sqrt{1+f_x^2+f_y^2}\, dx\,dy$$

so $\text{Area} = \iint_{\mathcal{R}} \sqrt{1+f_x^2+f_y^2}\, dx\,dy$

Example: (p. 1052)
Area cut from the plane $z = 2x+2y+1$
by the cylinder $x^2+y^2=1$



$f(x,y) = 2x+2y+1$

ans: 3π.