Name: SOLUTIONS
I.D. number

Math 2210-1
Exam 1
September 29, 2004

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

1a) Consider the following two lines, expressed parametrically:

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{\mathbf{q}}(s) = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Find the point (which exists!) at which these two lines intersect.

(12 points)

at intersection point

$x = 1 - t = 2 + s$
$y = -1 + 2t = -3 - 2s$
$z = 1 + t = -4 + s$

add E_1 & E_3:

$2 = -2 + 2s$
$4 = 2s$
$\boxed{2 = s}$

so $1 - t = 2 + 2$
$\boxed{t = -3}$

$$\text{point} = \vec{r}(-3) = \begin{bmatrix} 1+3 \\ -1-6 \\ 1-3 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -2 \end{bmatrix}$$

$$\text{check } \rho(2) = \begin{bmatrix} 2+2 \\ -3-4 \\ -4+2 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -2 \end{bmatrix} \checkmark$$

1b) Find an (implicit) equation of the form ax+by+cz=d for the plane containing the two lines in (1a). Notice that two parametric formulas contain enough information for you to do this problem without having completed (1a).

(12 points)

$$\langle a,b,c\rangle = \langle -1,2,1\rangle \times \langle 1,-2,1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \langle 4,2,0\rangle$$

direction vectors from the 2 lines

$$/\!/ \ \langle 2,1,0\rangle$$

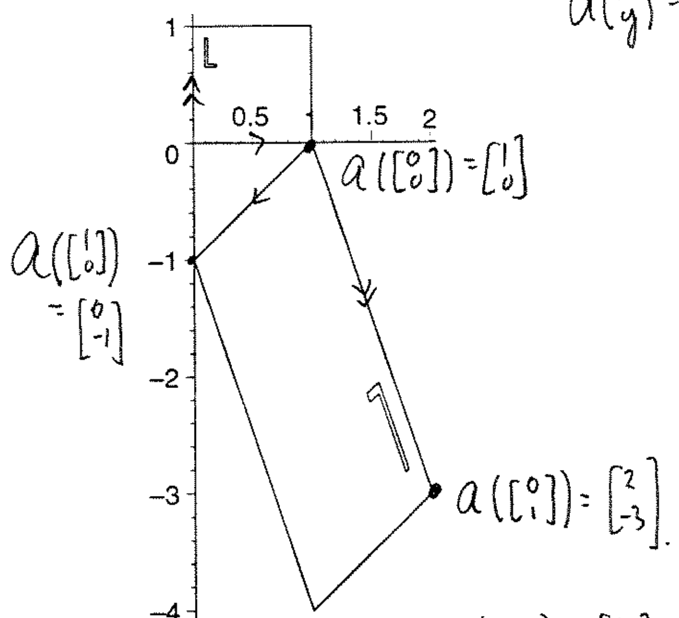
$$2x + y = 2(1) - 1 = 1$$

from point $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\boxed{2x + y = 1}$$

2) Consider an affine map from R^2 to R^2. Consider the following L-box diagram, which shows the unit "L-box" and where it is transformed to by our mystery affine map.

$$a\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}b_1\\b_2\end{bmatrix}$$



$$a\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$a\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = col_1(A) + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$$

2a) Find the formula for the mystery affine map.

(15 points)

$$\text{so } col_1(A) = \begin{bmatrix}0\\-1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$$

$$a\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = col_2(A) + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix}$$

$$\text{so } col_2(A) = \begin{bmatrix}2\\-3\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\-3\end{bmatrix}$$

$$\boxed{a\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \underbrace{\begin{bmatrix}-1 & 1\\-1 & -3\end{bmatrix}}_{\displaystyle \| \atop A}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}}$$

2b) What is the area of the image L-box?

(5 points)

$$\text{area} = |\det A| = 3 + 1 = 4$$

$$\left(\text{or area} = \begin{vmatrix}\rightarrow\\\twoheadrightarrow\end{vmatrix} = \begin{vmatrix}-1 & -1\\1 & -3\end{vmatrix} = 4\right)$$

$$\text{notice } \begin{vmatrix}a & b\\c & d\end{vmatrix} = \begin{vmatrix}a & c\\b & d\end{vmatrix}$$

3) Consider the parametric curve given by

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} \mathbf{e}^{t} \\ \mathbf{e}^{(-2\,t)} \end{bmatrix}$$

There is a sketch of part of the range of $\vec{\mathbf{r}}$ on the next page.

3a) Do a short computation to show that the range of $\vec{\mathbf{r}}$ lies on the graph $y = \frac{1}{x^2}$, in the x-y plane.

(4 points)

$x = e^{t}$
$y = e^{-2t}$ so $y = \frac{1}{(e^{t})^2} = \frac{1}{x^2}$

3b) Since the curvature of a curve is a property of the curve and not how it is parameterized, you can use the formula for the curvature of a graph $y = f(x)$ to compute the curvature at the image point (1,1). Do this, using the formula displayed on the blackboard.

(6 points)

$$\kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$$

$f(x) = x^{-2}$
$f'(x) = -2x^{-3}$
$f''(x) = 6x^{-4}$

$f''(1) = 6$
$f'(1) = 2$

$$\kappa = \frac{6}{(1+4)^{3/2}} = \frac{6}{5\sqrt{5}} \quad \left(= \frac{6}{5^{3/2}} \right)$$

3c) Returning to the parameterized version of the curve, compute $\vec{\mathbf{r}}\,'(t)$ and $\vec{\mathbf{r}}\,''(t)$.

(4 points)

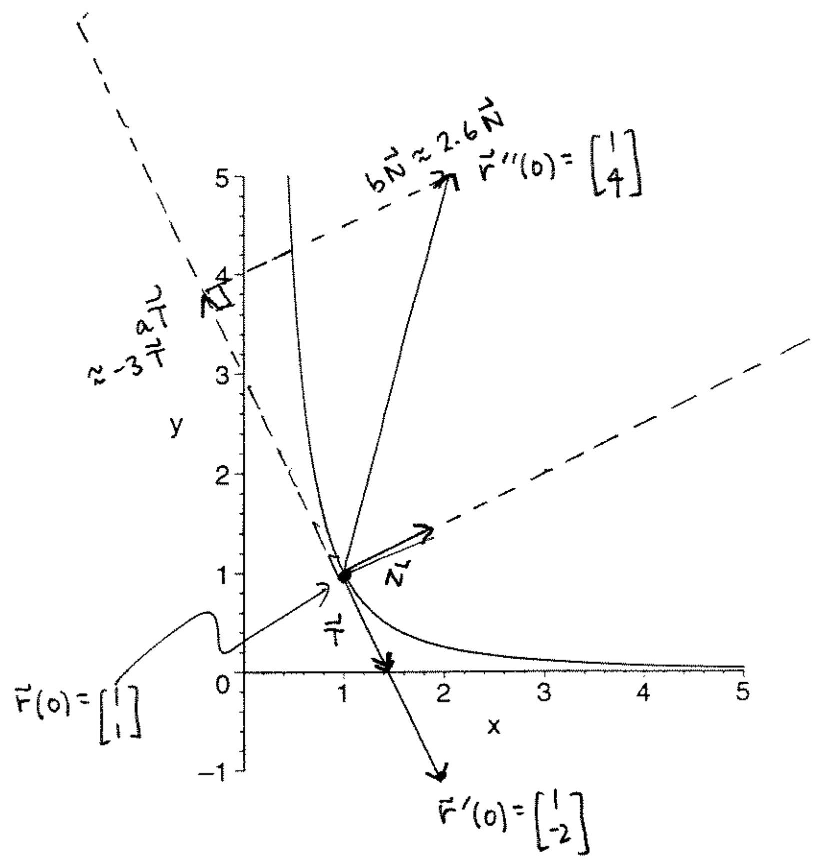
$$\vec{r}\,'(t) = \begin{bmatrix} e^{t} \\ -2e^{-2t} \end{bmatrix}$$

$$\vec{r}\,''(t) = \begin{bmatrix} e^{t} \\ 4e^{-2t} \end{bmatrix}$$

for next page, $\vec{r}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{r}\,'(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\vec{r}\,''(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$



3d) Label the point with position vector $\vec{\mathbf{r}}(0)$ into the picture above. Compute, and accurately draw the vectors $\vec{\mathbf{r}}\,'(0)$ and $\vec{\mathbf{r}}\,''(0)$ into the picture, in the appropriate location(s). Also, compute and draw in the unit tangent and normal vectors, $\vec{T}$ and $\vec{N}$, when $t = 0$. (Hint: you can find $\vec{N}$ easily because it is perpendicular to $\vec{T}$ and the curve lies in a plane.)

(10 points)

$\vec{r}'(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ so $\vec{T}(0) = \vec{T} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$; $\vec{N} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ since $\vec{N} \perp \vec{T}$ & pts in direction of bending.

$\vec{r}''(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

3e) Use your sketch, and an appropriate right triangle which you add to it, to crudely estimate (accuracy within 0.5 suffices) scalars a and b so that

$$\vec{\mathbf{r}}\,''(0) = a\vec{T} + b\vec{N}$$

(4 points)

using my ruler, I estimate $a \approx -3$
$b \approx 2.6$

(see figure)

Do ONE of (3f), (3g). If you try both ways, indicate which one you would like graded:

3f) You will now find the exact values of a and b: First, explain why the (3e) equation implies that

$$a = \vec{\mathbf{r}}''(0) \cdot \vec{T}$$
$$b = \vec{\mathbf{r}}''(0) \cdot \vec{N}$$

Then compute the dot products to recover a and b. (Hopefully your exact values for a and b are consistent with your numerical estimates from (3e).

(12 points)

$$\vec{r}''(0) = a\vec{T} + b\vec{N}$$

dot both sides with $\vec{T}$:

$$\vec{r}''(0)\cdot\vec{T} = (a\vec{T} + b\vec{N})\cdot\vec{T} = a\cdot 1 + b\cdot 0 = a$$

If I dot both sides with $\vec{N}$, get

$$\vec{r}''(0)\cdot\vec{N} = b$$

or

$$a = \mathrm{comp}_{\vec{T}}\,\vec{r}''(0) = \vec{r}''(0)\cdot\vec{T}$$
$$b = \mathrm{comp}_{\vec{N}}\,\vec{r}''(0) = \vec{r}''(0)\cdot\vec{N}$$

so

$$a = \begin{bmatrix}1\\4\end{bmatrix}\cdot\frac{1}{\sqrt5}\begin{bmatrix}1\\-2\end{bmatrix} = \frac{-7}{\sqrt5} \approx -3.1$$
$$b = \begin{bmatrix}1\\4\end{bmatrix}\cdot\frac{1}{\sqrt5}\begin{bmatrix}2\\1\end{bmatrix} = \frac{6}{\sqrt5} \approx 2.5$$

agrees!

OR

3g) Another way to get a and b is to use the acceleration decomposition formula

$$\vec{\mathbf{r}}''(t) = \left(\frac{d}{dt}v(t)\right)\vec{T} + \kappa v^2\vec{N}$$

where the scalar v represents speed and κ is the curvature. Using your curvature computation in (3b), and computing the speed and it's derivative at t=0, compute the values for a and b in the equation from (3e).

(12 points)

$$\left.\begin{array}{l} v(0) = |\vec{r}'(0)| = \sqrt5 \\ \kappa = \frac{6}{5\sqrt5}\end{array}\right) \text{ so } b = \kappa v^2 = \frac{6}{5\sqrt5}\cdot 5 = \frac{6}{\sqrt5}$$

$$v(t) = |\vec{r}'(t)| = \left(e^{2t} + 4e^{-4t}\right)^{1/2}$$
$$v'(t) = \frac12\left(e^{2t} + 4e^{-4t}\right)^{-1/2}\left(2e^{2t} - 16e^{-4t}\right)$$
$$v'(0) = \frac12(5)^{-1/2}(-14) = \frac{-7}{\sqrt5} = a$$

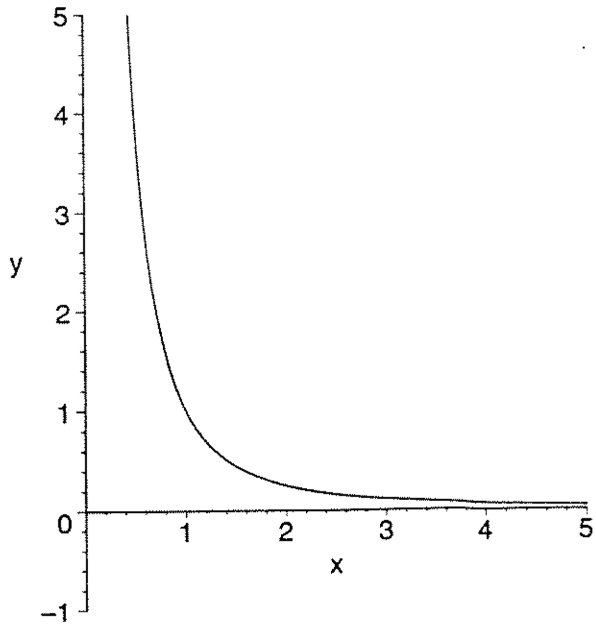
Remark (A 3rd way) (which some people did on 3e, not understanding my intensions)

This is really a system of 2 eqtns in the unknowns a, b:

$$\begin{bmatrix}1\\4\end{bmatrix} = a\begin{bmatrix}1/\sqrt5\\-2/\sqrt5\end{bmatrix} + b\begin{bmatrix}2/\sqrt5\\+1/\sqrt5\end{bmatrix} = \underbrace{\begin{bmatrix}1/\sqrt5 & 2/\sqrt5\\-2/\sqrt5 & +1/\sqrt5\end{bmatrix}}_{A}\begin{bmatrix}a\\b\end{bmatrix}$$

has soln

$$\begin{bmatrix}a\\b\end{bmatrix} = A^{-1}\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}1/\sqrt5 & -2/\sqrt5\\2/\sqrt5 & +1/\sqrt5\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}-7/\sqrt5\\6/\sqrt5\end{bmatrix}$$



(spare picture and space)