Math 2210-1
Review Sheet
12/8/04

Final exam is comprehensive. One 4×6" index card allowed.
Approximate percentages by chapter:

15 : 25% (15.1–15.4 only)
14 : 30%
13 : 30%
12 : 15%
11 : has the tools required in chapters 12–15, but no problems will be only from this chapter

Review session Fri 12/10
9:40–10:30, in regular classroom.

Exam week office hours
M: 9:40–10:30, 12:55–1:45
T: 9:40–10:30
W: 9:40–10:30

Course diagram:

Basic tools (Chapter 11)
each tool is defined algebraically but has value because of its geometric significance
vectors
addition & scalar mult
magnitude
unit
dot product
cross product
matrices
determinants

Basic geometric objects
$\vec{F}: D \longrightarrow \mathbb{R}^m$
domain in $\mathbb{R}^n$ — range of F in $\mathbb{R}^m$

lines, curves, planes, surfaces
if the object is described parametrically ("explicitly")
that means you are describing it as the range of a function, e.g. $\vec{F}(t)$, $\vec{X}(u,v)$

if the object is described implicitly that means you are describing it as points in the domain of a function $\vec{F}$ so that $F(\vec{x}) = \vec{c}$, e.g. level curves & level surfaces, quadric surfaces
special coord systems
polar
cylindrical
spherical

Differentiability (affine approximation)
$$\vec{F}(\vec{x}+\vec{h}) = \vec{F}(\vec{x}) + \vec{F}'(x)\vec{h} + \text{error}$$
$\frac{|\text{error}|}{|h|} \to 0$ as $\vec{h} \to \vec{0}$

cases

$\vec{r}(t+\Delta t) \approx \vec{r}(t) + \vec{r}'(t)\Delta t$ curves
(also studied $\vec{r}''(t), \kappa, \vec{T}$)

$f(\vec{x}+\Delta\vec{x}) \approx f(\vec{x}) + \underbrace{\nabla f(\vec{x}) \cdot \Delta\vec{x}}_{\text{"}df\text{"}}$ scalar fcns
(showed this holds if f partials exist & are continuous)

$\vec{X}(u+\Delta u, v+\Delta v) \approx \vec{X}(u,v) + \vec{X}_u \Delta u + \vec{X}_v \Delta v$
$\vec{X}(u+\Delta u, v+\Delta v, w+\Delta w)$
$\approx X(u,v,w) + \vec{X}_u \Delta u + \vec{X}_v \Delta v + \vec{X}_w \Delta w$

Differentiability applications
$$D_{\vec{u}} f(x) = \lim_{t\to 0} \frac{f(\vec{x}+t\vec{u}) - f(\vec{x})}{t} = \nabla f(\vec{x}) \cdot \vec{u}$$

Chain rule (matrix form & scalar form)
Important special case
$$\frac{d}{dt}\left(f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\right)$$

If $\{\vec{x} \text{ s.t. } f(\vec{x}) = c\}$ is a level set, $\nabla f(\vec{x})$ is $\perp$ to it, at $\vec{x}$

Max-min problems
by constraint elimination
or by Lagrange multipliers

Basic integration

double or triple integrals over
rectangles or coordinate boxes

more complicated iterated integrals
domain $\leftrightarrow$ limits on iterated integrals

area, volume, mass, centroids, moments
(also for curves & surfaces below)

topics with * are listed for completeness but will not be on final exam

Integral definitions and/or substitutions & ALL based on affine (small scale) approximation

$\int_C f(\vec{x})\,ds \;:=\; \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt \qquad ds = |\vec{r}\,'(t)|\,dt$

$\int_C \vec{F}\cdot d\vec{x} = \int_C \vec{F}\cdot\vec{T}\,ds \;:=\; \int_a^b f(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt \qquad d\vec{x} = \vec{r}\,'(t)\,dt$

in $\mathbb{R}^2$, $\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\langle y', -x'\rangle\,dt$

in $\mathbb{R}^2$, $\vec{T}\,ds = d\vec{x} = \langle x'(t), y'(t)\rangle\,dt$
$\vec{n}\,ds = \langle y'(t), -x'(t)\rangle\,dt$

* $\iint_S f\,dA := \iint_R f(\vec{X}(u,v))\,|\vec{X}_u\times\vec{X}_v|\,du\,dv$

(only $f\equiv 1$, i.e. area on final exam)

$dA = |\vec{X}_u\times\vec{X}_v|\,du\,dv$
$= \sqrt{1+z_x^2+z_y^2}\,dx\,dy$ for $\vec{X}(x,y) = (x, y, z(x,y))$

* $\iint_S \vec{F}\cdot\vec{n}\,dA := \iint_R \vec{F}(\vec{X}(u,v))\cdot(\vec{X}_u\times\vec{X}_v)\,du\,dv$

* $\vec{n}\,dA = (\vec{X}_u\times\vec{X}_v)\,du\,dv$
$= \langle -z_x, -z_y, 1\rangle\,dx\,dy$ for graph

Change of variables

$\iint_{x\text{-}y \text{ region}} f\,dA = \iint_{u\text{-}v \text{ region}} f(\vec{X}(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$

$dA = \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}\,du\,dv$
$= \operatorname{abs}(\det(X'(u,v)))\,du\,dv$

$\iiint_{x\text{-}y\text{-}z \text{ region}} f\,dV = \iiint_{u\text{-}v\text{-}w \text{ region}} f(\vec{X}(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw$

$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw$

special cases:
polar $dA = r\,dr\,d\theta$
cyl $dV = r\,dr\,d\theta\,dz$
sph. $dV = \rho^2\sin\phi\,d\phi\,d\rho\,d\theta$

Fundamental Theorem of Calculus

$\int_a^b f'(x)\,dx = f(b) - f(a)$

conservative vector fields:
when $\vec{F} = \nabla f$, and how to find f if it exists
$\int_C \vec{F}\cdot d\vec{x} = f(B) - f(A)$ if $\vec{F} = \nabla f$, and why.

$\iint_R f_x\,dA = \int_{\partial R} f\,\hat{\imath}\cdot\hat{n}\,ds$, $\iint_R f_y\,dA = \int_{\partial R} f\,\hat{\jmath}\cdot\hat{n}\,ds$

* higher dimension analogs ($n=3$ etc)

Green's Thm: $\iint_R Q_x - P_y\,dA = \oint_{\partial R} P\,dx + Q\,dy$

$n=2$ Div Thm: $\iint_R \operatorname{div}\vec{F}\,dA = \int_{\partial R} \vec{F}\cdot\vec{n}\,ds$

* $n=3$ div thm $\iiint_R \operatorname{div}\vec{F} = \iint_{\partial R} \vec{F}\cdot\vec{n}\,dA$

* Stoke's thm $\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dA = \oint_{\partial S} \vec{F}\cdot d\vec{x}$

$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F}$
$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$

Sample final exam questions
(not inclusive!)

1 a) Sketch the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

b) Show that the curve $\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 3\sin t \end{bmatrix}$ lies on the ellipse

c) To your sketch in (a), add $\vec{r}(0)$, $\vec{r}\,'(0)$, $\vec{r}\,''(0)$, in appropriate locations (compute them first!)

d) If $\vec{r}(t)$ as above describes a particle motion, is the partical speeding up or slowing down (or neither) at $t=0$? Explain.

2. Consider the same ellipse as in problem 1.

a) Use the change of variables $u = x/2$, $v = y/3$ (actually its inverse) and the fact that the unit disk has area π, to show that the area inside the ellipse is 6π

b) Use Green's Theorem, with $\langle P, Q\rangle = \langle -y, x\rangle$, compute the line integral $\oint_{\partial R} P\,dx + Q\,dy$, where R is the region inside the ellipse (and ∂R is the ellipse)

and deduce from this computation that the area inside the ellipse is 6π

3. a) Compute $\int_0^4 \int_{y/2}^{\sqrt{y}} 4x\,dx\,dy$

b) Sketch the region of integration in 3a)

c) Express the integral in 3a) as an iterated integral with the order of integration reversed, and compute this integral

4. One of the vector fields below is a gradient vector field. Figure out which one it is, and for that field find a potential function f so that $\nabla f = \vec{F}$

(a) $\vec{F} = \langle \sin x + e^x \cos y,\ 3y^2 - e^x \sin y\rangle$

(b) $\vec{F} = \langle y, -x\rangle$

5. For the gradient field $\vec{F}$ in (4), let C be the line segment from $(0,0)$ to $(1,0)$

(a) Compute $\int_C \vec{F}\cdot d\vec{x}$ explicitly

(c) Use the potential function to compute $\int_C \vec{F}\cdot d\vec{x}$

6. For the polar coordinate transformation $\begin{bmatrix} x \\ y \end{bmatrix} = F(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

a) Compute the derivative matrix $F'(r,\theta)$

b) Explain how the $\det(F'(r,\theta))$ is related to the formula $dA = r\,dr\,d\theta$ is polar coord. substitution for double integrals

c) Use affine approximation to estimate $\begin{bmatrix} x \\ y \end{bmatrix}$ when $r = 2.1$, $\theta = -.1$ radians
(differential)

using the fact that $F(2,0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

7. You must build a rectangular shipping crate with volume 60 ft^3
Its sides cost \$1/ft^2, its top costs \$2/ft^2, and its bottom costs \$3/ft^2.
What dimensions would minize to total cost?

(a) Solve this by using the constraint to eliminate a variable.

(b) Resolve the problem with Lagrange multipliers.

8. A uniform wire with density 2 gm/cm $= 200$ gm/m is shaped like the semicircle $x^2+y^2=4$, $y>0$ (i.e. its radius is 2 m = 200 cm).

(a) Find the mass of the wire

(b) Find its centroid.

9. Find the volume of the region that lies inside the sphere $x^2+y^2+z^2=9$ but outside the cylinder $x^2+y^2=4$.

10. Compute the divergence and curl of
$\vec{F}(x,y,z) = \langle e^x\cos y,\ e^x\sin y + z,\ xy \rangle$