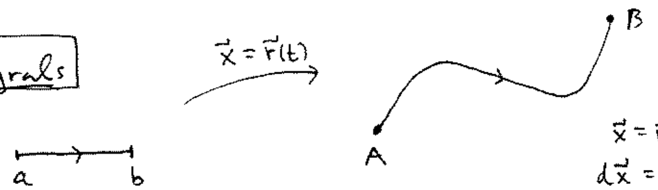
Math 2210-1
Friday 12/3
§15.4 (with ideas thru 15.7).

The multivariable Fundamental Theorem of Calculus (FTC) and its various manifestations.
(Green's, Stokes, Divergence (Gauss') Theorems).

The zoo of curve and surface integrals you need to visit and know:

curve integrals



$\vec{x} = \vec{r}(t)$
$d\vec{x} = \vec{r}\,'(t)\,dt$
$ds = |d\vec{x}| = |\vec{r}\,'(t)|\,dt$

(1) $\boxed{\int_C f(\vec{x})\,ds} := \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

(2) $\boxed{\int_C \vec{F}\cdot d\vec{x}} := \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$ $\quad\vec{T}$ unit tang. vect

"work"

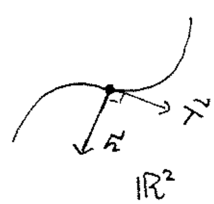
$= \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\,|\vec{r}\,'(t)|\,dt$

$= \boxed{\int_C (\vec{F}\cdot\vec{T})\,ds}$

(3) $n=2$ (plane curve)

"flux" $\boxed{\int_C (\vec{F}\cdot\vec{n})\,ds}$

the right-side normal



$\vec{T} = \frac{1}{|\vec{r}\,'(t)|}\langle x'(t), y'(t)\rangle$

$\vec{n} = \frac{1}{|\vec{r}\,'(t)|}\langle y'(t), -x'(t)\rangle$

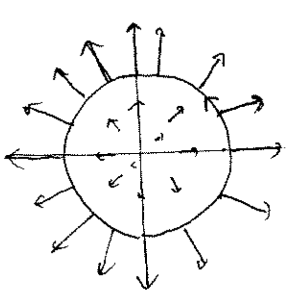
$= \int_a^b \vec{F}\cdot\langle y', -x'\rangle\,dt \qquad \vec{n}\,ds = \langle y', -x'\rangle$

examples

(1) $\vec{F} = \langle x, y\rangle$
C = unit circle
c.clckwise

$\oint_C (\vec{F}\cdot\vec{T})\,ds =$

$\int_C (\vec{F}\cdot\vec{n})\,ds =$



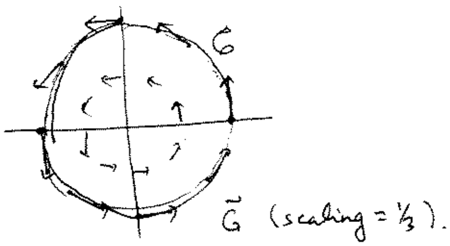
$\vec{F}$ (scaling $\approx \frac{1}{2}$)

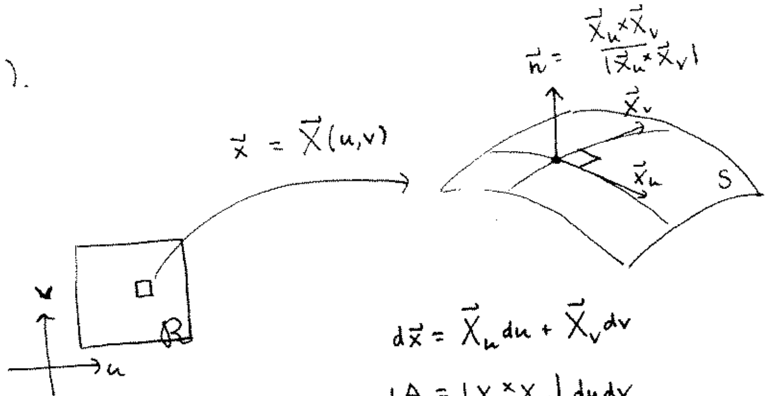
you can deduce these without explicit computation of integrals!

(2) $\vec{G} = \langle -y, x \rangle$ $\quad \oint_C \vec{F}\cdot\vec{T}\,ds =$

C = unit circle

$\int_C \vec{F}\cdot\vec{n}\,ds =$





Surface integrals

(1) $\iint_S f(\vec{x})\,dA$

$$= \iint_R f(\vec{X}(u,v))\,|\vec{X}_u \times \vec{X}_v|\,du\,dv$$

$d\vec{x} = \vec{X}_u\,du + \vec{X}_v\,dv$

$dA = |X_u \times X_v|\,du\,dv$

$\vec{n}$ (unit normal)

$$= \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}$$

$\vec{n}\,dA = (\vec{X}_u \times \vec{X}_v)\,du\,dv$

(2) $\iint_S (\vec{F}\cdot\vec{n})\,dA = \iint_R \vec{F}\cdot(\vec{X}_u \times \vec{X}_v)\,du\,dv$

flux integrals

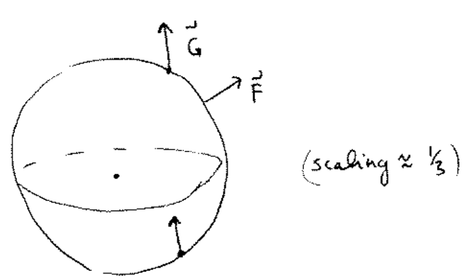
(2') $\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dA$

Example 3 $\vec{F} = \langle x, y, z \rangle$

S = unit sphere at origin

$\vec{n}$ = exterior normal (unit vect).

$\iint_S (\vec{F}\cdot\vec{n})\,dA =$

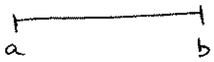


Example 4 $\vec{G} = \langle 0, 0, 1 \rangle$

$\iint_S (\vec{G}\cdot\vec{n})\,dA =$

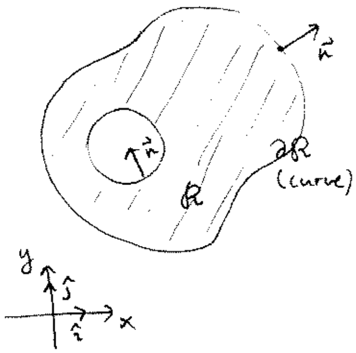
Fundamental Theorem of Calculus
(multivariable).

✓ $n=1$



$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$n=2$

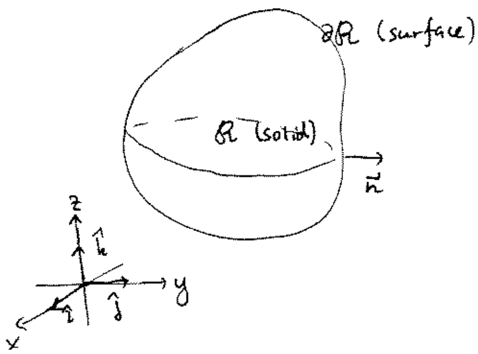


$$\iint_R \frac{\partial f}{\partial x}\,dA = \int_{\partial R} f\,\hat{i}\cdot\vec{n}\,ds$$

$$\iint_R \frac{\partial f}{\partial y}\,dA = \int_{\partial R} f\,\hat{j}\cdot\hat{n}\,ds$$

$\hat{i} = \langle 1, 0\rangle$
$f\,\hat{i}\cdot\hat{n}$
$= \langle f, 0\rangle\cdot\vec{n}$
$\vec{n}$ exterior unit normal

$n=3$



$$\iiint_R f_x\,dV = \iint_{\partial R} f\,\hat{i}\cdot\vec{n}$$

$$\iiint_R f_y\,dV = \iint_{\partial R} f\,\hat{j}\cdot\vec{n}$$

$$\iiint_R f_z\,dV = \iint_{\partial R} f\,\hat{k}\cdot\vec{n}$$

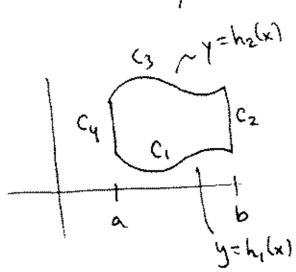
$\vec{n}$ exterior unit normal.

"proof" $n=2$

$$\iint_R \frac{\partial f}{\partial y}\,dA = \int_{\partial R} f\,\hat{j}\cdot\hat{n}\,ds$$

($\iint_R \frac{\partial f}{\partial x}\,dA = \int_{\partial R} f\,\hat{i}\cdot\vec{n}\,ds$ is analogous).

case I: vertically simple region : $\boxed{\int_a^b \left(\int_{h_1(x)}^{h_2(x)} \frac{\partial f}{\partial y}(x,y)\,dy\right) dx}$



$$= \int_a^b f(x,y)\Big]_{y=h_1(x)}^{h_2(x)}\,dx$$

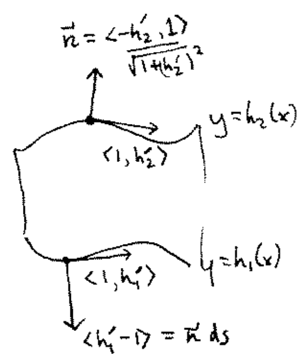
$$= \int_a^b f(x,h_2(x)) - f(x,h_1(x))\,dx$$

$$= \underbrace{\int_a^b f(x,h_2(x))\,dx}_{} - \underbrace{\int_a^b f(x,h_1(x))\,dx}_{} + \underbrace{0}_{} + \underbrace{0}_{}$$

$$(*)\ \int_{C_3} f\,\hat{j}\cdot\vec{n}\,ds + \int_{C_1} f\,\hat{j}\cdot\vec{n}\,ds + \int_{C_2} f\,\hat{j}\cdot\vec{n}\,ds + \int_{C_4} f\,\hat{j}\cdot\vec{n}\,ds = \boxed{\int_{\partial R} f\,\hat{j}\cdot\hat{n}\,ds}$$

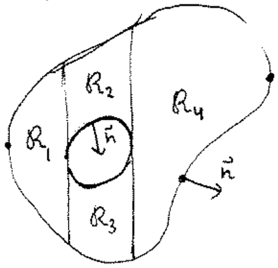
$\vec{n}\,ds = \langle -h_2', 1\rangle$

(*) details

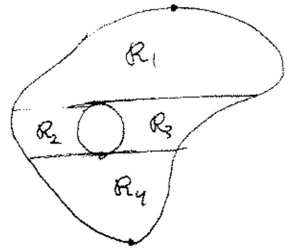


$n=2$ general region: $\displaystyle\iint_R \frac{\partial f}{\partial y}\, dA = \int_{\partial R} f\,\hat{j}\cdot\hat{n}\, ds$

Decompose & add



$$\iint_R \frac{\partial f}{\partial x}\, dA = \int_{\partial R} f\,\hat{i}\cdot\hat{n}\, ds$$



$n=3$ (to be filled in) Monday

For scientific reasons, you usually see FTC in forms with divergence & curl operators rather than simple partial derivatives.

$n=2$ <u>divergence theorem</u>:

$$\boxed{\iint_R \vec{\nabla}\cdot\vec{F}\,dA = \int_{\partial R} \vec{F}\cdot\vec{n}\,ds}$$

<u>pf</u> let $\vec{F} = \langle R, S\rangle$

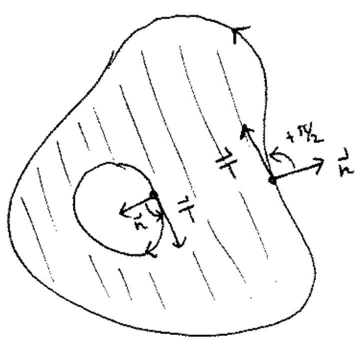
$$\left.\begin{aligned} &\iint_R R_x\,dA = \int_{\partial R} R\,\hat{\imath}\cdot\vec{n}\,ds \\ + &\iint_R S_y\,dA = \int_{\partial R} S\,\hat{\jmath}\cdot\vec{n}\,ds \end{aligned}\right\} \text{page 3 FTC}$$

$$\iint_R \langle \partial_x, \partial_y\rangle\cdot\langle R, S\rangle\,dA = \int_{\partial R} (R\hat{\imath} + S\hat{\jmath})\cdot\vec{n}\,ds \quad \blacksquare$$

$n=2$ <u>Green's theorem</u>:

$$\boxed{\iint_R (Q_x - P_y)\,dA = \oint_{\partial R} \langle P, Q\rangle\cdot\vec{T}\,ds = \oint_{\partial R} P\,dx + Q\,dy}$$

$\uparrow$
scalar curl of $\vec{F} = \langle P, Q\rangle$



pf: $\displaystyle \iint_R (Q_x - P_y)\,dA = \iint_R \langle\partial_x, \partial_y\rangle\cdot\langle Q, -P\rangle\,dA$

$\displaystyle = \int_{\partial R} \langle Q, -P\rangle\cdot\vec{n}\,ds$ by div thm

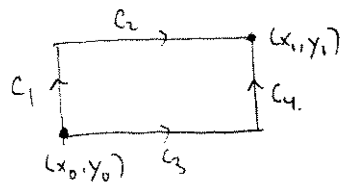
if $\vec{T} = \langle a, b\rangle$
$\vec{n} = \langle b, -a\rangle$

$\langle Q, -P\rangle\cdot\langle b, -a\rangle$
$= \langle P, Q\rangle\cdot\langle a, b\rangle$!

$\displaystyle = \int_{\partial R} \langle P, Q\rangle\cdot\vec{T}\,ds \quad \blacksquare$

<u>check examples 1 & 2 from page 1, 2</u>:

$n=2$ Green's theorem explains why the condition $Q_x = P_y$ (in a rectangle domain) implies $\langle P, Q\rangle = \nabla f$; a point left open on Wed notes



$$f(x,y) := \int_{C_1 \cup C_2} P\,dx + Q\,dy$$

$$= \int_{C_3 \cup C_4} P\,dx + Q\,dy$$

since $\displaystyle\int_{C_3} (\ ) + \int_{C_4} - \int_{C_2} - \int_{C_1} P\,dx + Q\,dy \overset{\text{Green's}}{=} \iint_R Q_x - P_y \, dA = 0$