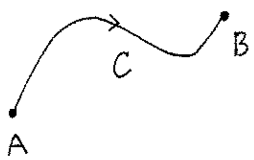
Math 2210-1
Wed 12/1

§ 15.3 : Independence of path for line integrals.

HW due 12/8
Last assignment hurray!

§ 15.3 3, 5, 17, 21, 28
§ 15.4 1, 6 ← also, compute the actual line integral in 6
7, 17, 21
§ 15.5 1, 13, 15, 27
15.6 1, 3, 6
15.7 1, 11, 17

Question:



If C is a curve connecting A to B
For what vector fields $\vec{F}$ does the line integral

$$\int_C \vec{F}\cdot d\vec{x}$$

only depend on the initial pt A and terminal point B, but not on the particular path C?
In this case, the line integrals for $\vec{F}$ are called path independent

Answer : Line integrals for $\vec{F}$ are path independent if and only if, $\vec{F}$ is a gradient vector field,

$$\vec{F} = \nabla f$$

½
Why : We explained half the answer on Monday:
If $\vec{F} = \nabla f$, then $\int_C \vec{F}\cdot d\vec{x} = f(B) - f(A)$

(since $\int_C \nabla f \cdot d\vec{x} = \int_a^b \underbrace{\nabla f(\vec{r}(t))\cdot \vec{r}\,'(t)}_{\frac{d}{dt} f(\vec{r}(t)) \text{ by chain rule}}\,dt$, where $\vec{r}:[a,b] \to \mathbb{R}^n$ parameterizes C, $\vec{r}(a)=A$, $\vec{r}(b)=B$

$$= f(\vec{r}(t))\Big]_a^b \quad \text{by 1210 F.T.C.}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(B) - f(A).$$

Example 1: HW collected today, § 15.2 #15: $\vec{F} = \langle yz^2, xz^2, 2xyz\rangle$

$A = (-1,2,-2)$
$B = (1,5,2)$

in HW, your path from A to B followed three coord. direction line segments, and you got $\int_C \vec{F}\cdot d\vec{x} = 28$

Example 1 cont'd

(a) Compute $\int_C \vec{F}\cdot d\vec{x}$ where C is a straight segment connecting $(-1,2,-2)$ to $(1,5,2)$

verify that this line integral also yields 28.

$$\vec{r}(t) = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad 0 \le t \le 1$$

$$\int_C \vec{F}\cdot d\vec{x} = \int_0^1 \Big\langle \underbrace{(2+3t)}_{y}\underbrace{(-2+4t)^2}_{z^2},\ \underbrace{(-1+2t)}_{x}\underbrace{(-2+4t)^2}_{z^2},\ 2\underbrace{(-1+2t)}_{x}\underbrace{(2+3t)}_{y}\underbrace{(-2+4t)}_{z} \Big\rangle \cdot \underbrace{\langle 2,3,4\rangle}_{\vec{r}\,'(t)}\, dt$$

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> x:=-1+2*t:
  y:=2+3*t:
  z:=-2+4*t:
  int(2*y*z^2+3*x*z^2+4*2*x*y*z,t=0..1);
                                        28
>
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(b) Notice $\vec{F} = \nabla f$

where $f(x,y,z) = xyz^2$

Thus $\int_C \vec{F}\cdot d\vec{x} = f(B) - f(A)$

verify this value is also 28.

Is this way easier?

<u>other ½ why</u> :

If integrals are path independent, then you can use them to find (define) f, with $\nabla f = \vec{F}$.

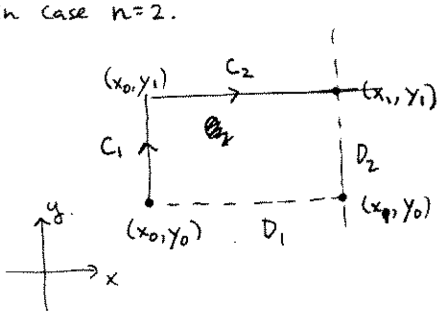
Define $f(\vec{x}) = \int_C \vec{F} \cdot d\vec{x}$, where C connects a fixed base point $\vec{x}_0$ to the varying point $\vec{x}$.

Then $\nabla f = \vec{F}$

(C is arbitrary since we <u>assume</u> $\vec{F}$ yields path independent line integrals

proof : in case $n=2$.

$\vec{F} = \langle P, Q \rangle$



~~other~~

$$f(x, y_1) = \int_{C_1} Q\,dy + \int_{x_0}^{x} P(t, y_1)\,dt$$

so $\frac{\partial f}{\partial x}(x_1, y_1) = 0 + P(x_1, y_1)$. ✓.

(by 1210 FTC.).

$$f(x_1, y) = \int_{D_1} P\,dx + \int_{y_0}^{y} Q(x_1, t)\,dt$$

so $\frac{\partial f}{\partial y}(x_1, y_1) = Q(x_1, y_1)$ ✓.

<u>And</u> : If the vector field $\vec{F}$ is differentiable everywhere (or in a rectangular domain),

then $\vec{F}$ yields path independent line integrals exactly when

$n=2$, $\vec{F} = \langle P, Q \rangle$, $Q_x = P_y$ (would be saying $(f_y)_x = (f_x)_y$)

$n=3$, $\vec{F} = \langle P, Q, R \rangle$, $\vec{\nabla} \times \vec{F} = \vec{0}$ (would be saying all cross partials are equal).

follows from Green's thm § 15.4

follows from Stoke's thm § 15.~~6~~7

<u>example 2</u> : Show $\vec{F} = \langle 2x+y,\ x^2-y^2 \rangle$ is <u>not</u> a gradient field.

<u>example 3</u> : Show $\vec{F} = \langle 6xy - y^3,\ 4y + 3x^2 - 3xy^2 \rangle$ <u>is</u> a gradient field, and find f so that $\nabla f = \vec{F}$

(p 1088–1089 text)

<u>method 1</u> : $f(x,y) = \displaystyle\int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{x}$

<u>method 2</u> : "antidifferentiation".

Physics application

$\vec{F}$ = force field.

$\mathcal{C}$ connects A to B

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{x} \;:=\; \underset{W}{\text{work done by}}\ \vec{F} \text{ to move object from A to B along } \mathcal{C}$$

$$PE := -\int_C \vec{F}\cdot d\vec{x} \quad = \text{work done by object} \qquad \text{(Potential energy)}$$

$$KE := \tfrac{1}{2}mv^2 \quad , \quad v = |\vec{r}\,'(t)| \quad , \quad \text{Kinetic energy.}$$

If $\vec{F}$ is a $\underset{\vec{F}=\nabla f}{\text{gradient}}$ vector field, it is called conservative, because

$$\text{Newton } (\vec{F} = m\vec{r}\,''(t)) \implies PE + KE \equiv \text{constant} \quad \text{for partical motion.}$$

proof

$$PE := -\int_{\vec{r}(0)}^{\vec{r}(t)} \vec{F}\cdot d\vec{x} \;=\; f(\vec{r}(0)) - f(\vec{r}(t))$$

$$KE = \tfrac{1}{2}\, m\, \vec{r}\,'(t)\cdot\vec{r}\,'(t)$$

$$\frac{d}{dt}(KE + PE) = \frac{d}{dt}\Big(\tfrac{1}{2}m\vec{r}\,'\cdot\vec{r}\,' + f(\vec{r}(0)) - f(\vec{r}(t))\Big)$$

$$= \underset{\text{(product rule)}}{m\vec{r}\,'\cdot\vec{r}\,''} - \underset{\text{(chain rule)}}{\nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)}$$

$$= \vec{r}\,'\cdot\Big[\underbrace{m\vec{r}\,'' - \vec{F}(\vec{r}(t))}_{=0 \text{ by Newton!}}\Big]$$

examples

uniform grav. field

$$\vec{F} = \begin{bmatrix} 0 \\ 0 \\ -gm \end{bmatrix} = \nabla(-mgz)$$

$$PE = mgz \quad (+\text{const})$$

inverse square law

$$\vec{F} = \left(-\frac{c}{|\vec{r}|^3}\right)\vec{r}$$

$$f = \frac{c}{|\vec{r}|}$$

$$PE = -\frac{c}{|\vec{r}|} \qquad (+\text{ const.})$$