Math 2210-1
Monday 30 Aug.

HW due Wed!
Problem session tomorrow
9:40–10:30 LCB 121

hint: on exercise III, law of cosines, you only need to use Pythag thm once, not twice as stated.

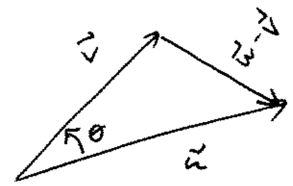
Finish Friday notes:

- sphere example page 3
- computational example page 3
- Do dot product page 4

$$\left.\begin{array}{l}\vec{u} = \langle u_1, u_2, u_3\rangle \\ \vec{v} = \langle v_1, v_2, v_3\rangle\end{array}\right\} \vec{u}\cdot\vec{v} := u_1v_1 + u_2v_2 + u_3v_3$$

details of geometric property

$$\boxed{\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta}$$



proof:

By law of cosines (see Hw)

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$

By dot product properties

$$\begin{aligned}|\vec{u}-\vec{v}|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\ &= \vec{u}\cdot(\vec{u}-\vec{v}) - \vec{v}\cdot(\vec{u}-\vec{v}) \\ &= \vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\ &= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v}\end{aligned}$$

Since these eqtns both express $|\vec{u}-\vec{v}|^2$, deduce

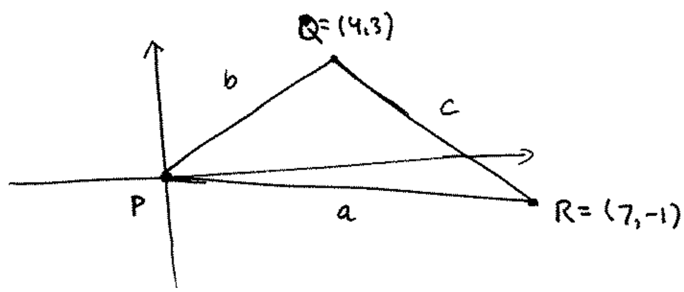
$$-2|\vec{u}||\vec{v}|\cos\theta = -2\vec{u}\cdot\vec{v}$$

i.e.

$$|\vec{u}||\vec{v}|\cos\theta = \vec{u}\cdot\vec{v} \quad \blacksquare$$

Examples: is $\triangle PQR$ a right triangle?

$P = (0,0)$
$Q = (4,3)$
$R = (7,-1)$

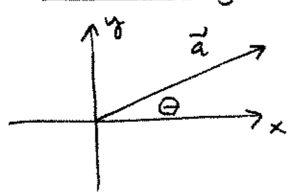


$\vec{PQ} = \langle 4,3\rangle$
$\vec{PR} = \langle 7,-1\rangle$
$\vec{QR} = \langle 3,-4\rangle$

$$\boxed{\text{Deduce: } \vec{u}\perp\vec{v} \text{ if and only if } \vec{u}\cdot\vec{v} = 0.}$$

$\uparrow$ perpendicular

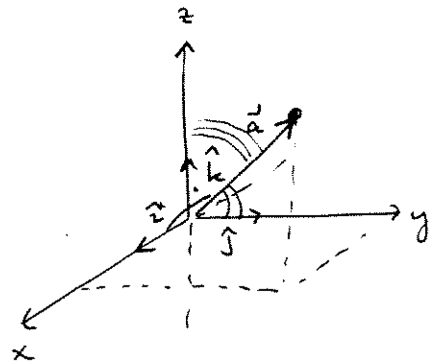
direction angles



$\vec{a} = \langle a_1, a_2 \rangle$

$$\cos(\theta) = \frac{\vec{a}\cdot\hat{\imath}}{|\vec{a}|\cdot 1} = \frac{a_1}{|\vec{a}|}$$

$\mathbb{R}^3$ similar $\rightsquigarrow$



$\measuredangle\, \hat{\imath}, \vec{a} := \alpha$ $\qquad \cos\alpha = \dfrac{\vec{a}\cdot\hat{\imath}}{|\vec{a}|\cdot 1} = \dfrac{a_1}{|\vec{a}|}$

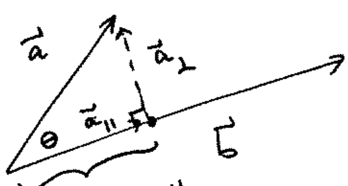
$\measuredangle\, \hat{\jmath}, \vec{a} := \beta$ $\qquad \cos\beta = \dfrac{a_2}{|\vec{a}|}$

$\measuredangle\, k, \vec{a} := \gamma$ $\qquad \cos\gamma = \dfrac{a_3}{|\vec{a}|}$

components and vector decomposition

$\downarrow$

generalizes idea of x, y $(\hat{\imath}, \hat{\jmath})$ components
or x, y, z $(\hat{\imath}, \hat{\jmath}, \hat{k})$ components of vectors.



length

$\text{comp}_{\vec{b}}\,\vec{a}$ (component of $\vec{a}$ in direction of $\vec{b}$)

$:= |\vec{a}|\cos\theta$ = signed length of projection of $\vec{a}$ onto $\vec{b}$

+ if $\theta < \pi/2$
− if $\theta > \pi/2$

$$= |\vec{a}|\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

$$= \vec{a}\cdot\frac{\vec{b}}{|\vec{b}|}$$

(the dot product of $\vec{a}$ with the unit vector in the $\vec{b}$ direction)

then write

$\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$:

$$a_{\parallel} = \left(\text{comp}_{\vec{b}}\,\vec{a}\right)\frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\,\vec{b}$$

$$a_{\perp} = \vec{a} - a_{\parallel}$$

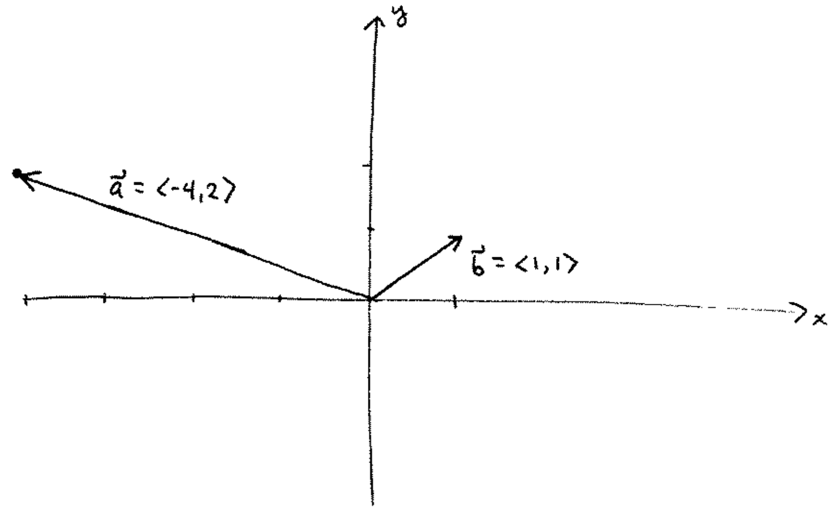
example

$\vec a = \langle -4,2\rangle$

$\vec b = \langle 1,1\rangle$

find $\text{comp}_{\vec b}\vec a$, $\vec a_{\parallel}$, $\vec a_{\perp}$

draw a picture

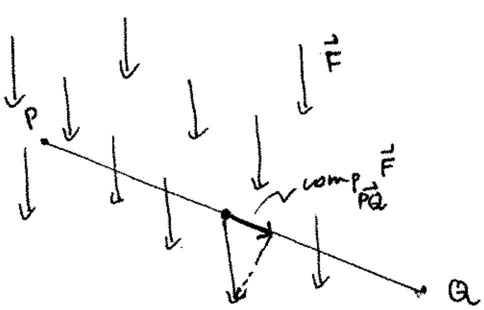


example : "Work" in physics. (see e.g. 11.2 # 56,57)

If $\vec F$ is a constant force field, and if an object moves (in a straight line) from P to Q, then work done by the force field is (opposite of work done by the object)

$$W := \vec F \cdot \vec{PQ}$$
$$= \left(\text{comp}_{\vec{PQ}}\vec F\right)|\vec{PQ}|$$

this generalizes examples you did in Calculus in which the force field was ~~the~~ in the direction of motion.



e.g. if $\vec F = -\overbrace{m}^{\text{object mass}}\overbrace{g}^{9.8\ m/sec^2}\vec k$

$$W = \vec F\cdot\vec{PQ} = (-mg\vec k)\cdot(\overbrace{h\vec k}^{\text{height}} + \overbrace{\vec T}^{\text{horizontal}})$$
$$= -mgh.$$

§ 11.3

determinants & cross product : intro

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} := a_1b_2 - b_1a_2$$ 2×2 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} := +a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

3×3 det - expansion across top row. Note the 2×2 dets are obtained by crossing out the row and column of the entry coefficient

- you can expand across any row or down any column, ~~g~~ using the checkerboard template of ±'s.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

e.g. down column 3, you get $a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

- in all cases you end up with

$$\boxed{\begin{aligned} &a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \\ &\quad - c_1b_2a_3 - c_2b_3a_1 - c_3b_1a_2 \end{aligned}}$$

<u>cross product</u>

$\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

then $\vec{a}\times\vec{b} \left(\text{"="} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \text{"="} \right) = \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$= \langle a_2b_3 - b_2a_3,\ b_1a_3 - a_1b_3,\ a_1b_2 - b_1a_2 \rangle$

<u>example</u>

$\vec{a} = \langle 2, 1, 0 \rangle$
$\vec{b} = \langle -2, 4, 0 \rangle$

compute $\vec{a}\times\vec{b}$

verify (for this example)

(1) $\vec{a}\times\vec{b} \perp \vec{a}, \vec{b}$, <u>and</u> $\{\vec{a}, \vec{b}, \vec{a}\times\vec{b}\}$ are right-handed.

(2) $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$