

Math 1210-001
Wednesday Mar 2
WEB L112

Review day for exam 2. We can discuss any WebWork or lab problems from this week that you'd like hints for, or that confuse you. There are also worked examples in these notes, from earlier sections.

- Exam is this Friday March 4, 9:35 AM-10:35 AM WEB L112.
(We start 5 minutes early and end 5 minutes late, compared to usual meeting pattern.)
- No notes or collaboration allowed. You may use a scientific calculator (the sort that costs around \$15, e.g. a TI 30X). Symbolic answers will be accepted, so a calculator is not needed.
- The exam will be organized into 5 or so problems, with most problems having several parts.
- Your lab sessions tomorrow (Thursday) will be organized like the usual labs, except you'll be working on review problems - you'll be able to focus on the areas that concern you the most, in the lab. I'll post solutions to those review problems and to the lab you're handing in, by noon on Thursday. (For next week, your homework will consist of a relatively short WebWork assignment on sections 3.4-3.5; some of the questions will be separated out for you to hand in as free-response work at the start of next week's lab.)

The material you're responsible for on exam 2 is Chapter 2.4-3.3. In addition, we've been using first and second derivative information to draw various graphs, and there will be a graphing problem on the exam - of the sort we've focused on so far. The material for this exam implicitly requires what you've learned earlier in the course as well. Stars next to a topic means that there will be at least one problem that tests that section material directly. (A single problem can test more than one topic at a time.)

2.4 Derivatives of trig functions (know these, they may show up in other problems even though they won't have a problem of their own).

2.5 Chain rule (our most important differentiation rule - may or may not be tested directly, but is the key to implicit differentiation and related rates applications, and can show up in all other applications as well.)

2.6 Higher order derivatives (especially second derivatives, and what they measure)

2.7 Implicit differentiation *

2.8 Related rates * (You will be provided with the geometry cut-out page from the text, and should also be able to use similar triangles.)

2.9 Differentials and approximation *

3.1 Maxima and minima *

3.2 Monotonicity and concavity *

3.3 Local extrema *

(Graphing functions using Calculus * The exam graph will be a relatively low degree polynomial or rational function, since those are the sorts of graphs we've focused on the most so far. Graphing problems can integrate many ideas from 3.1-3.3 at once.)

The following pages contain a synopsis of what we've covered in these sections. This is mostly material for you to go over carefully on your own, as you organize your studying; we can discuss some of it in class as appropriate.

You should know differentiation rules and basic trig identities:

$$D_x(x^n) = n x^{n-1}, n \in \mathbb{Z}. \quad (\text{power rule; includes the three special cases above.})$$

$$D_x(f(x) + g(x)) = D_x(f(x)) + D_x(g(x)) \quad (\text{sum rule})$$

$$D_x(kf(x)) = k D_x(f(x)) \text{ if } k \text{ is a constant.} \quad (\text{constant multiple rule})$$

$$D_x(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \quad (\text{product rule})$$

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}. \quad (\text{quotient rule})$$

$$D_x(f(g(x))) = f'(g(x)) \cdot g'(x) \quad (\text{chain rule})$$

$$D_x(f(g(h(x)))) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) \quad (\text{extended version})$$

$$D_x(\sin(x)) = \cos(x)$$

$$D_x(\cos(x)) = -\sin(x)$$

$$D_x \tan(x) = \sec^2(x)$$

$$D_x \cot(x) = -\csc^2(x)$$

$$D_x \sec(x) = \sec(x)\tan(x)$$

$$D_x \csc(x) = -\csc(x)\cot(x)$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha + \beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta).$$

2.7 Implicit differentiation: The solution set (and graph) of an equation in two variables x, y may implicitly define y as a function of x , near a point (x_0, y_0) on the graph. By differentiating the resulting identity with respect to x , and using the chain rule, one may solve for y' in terms of x, y .

Example: Find y' (meaning $y'(x)$) if x, y are related by

$$x^2 y^2 + \cos(3y) = 1.$$

Solution: Treat y as a function $y(x)$, and take D_x of the identity:

$$\begin{aligned} 2xy + x^2 2yy' - \sin(3y) 3y' &= 0 \\ (2x^2y - 3\sin(3y))y' &= -2xy \\ y' &= -\frac{2xy}{(2x^2y - 3\sin(3y))}. \end{aligned}$$

Extensions: Find y' at specific points (x_0, y_0) ; find equations of tangent lines at those points.

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2.8 Related rates: One or more identities relate several time-varying functions. (You may have to find these identities.) Use these identities and their derivatives with respect to time, to deduce how fast one of the functions is changing, if you have enough information about how fast the others are changing and their function values, at some instant in time.

Example: Water leaking onto the floor makes a growing circular pool. If the radius of the pool is increasing at a rate of $4 \frac{cm}{min}$ when the radius is $30 cm$, how fast is the area of the pool changing?

Solution:

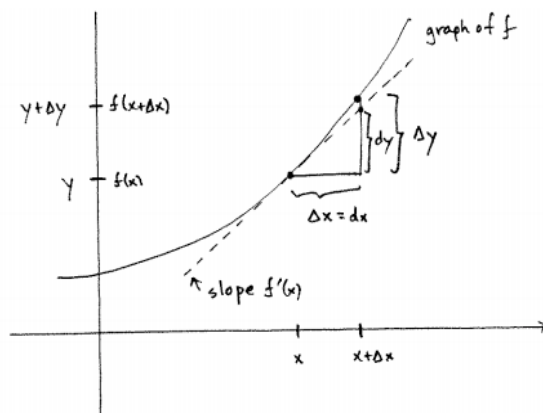
- We could draw a picture...
- The relevant functions of time that are changing, are the circle radius $r(t)$ and the enclosed area $A(t)$.
- We are told that when $r = 30$, $r'(t) = 4$ (and we know the units of each).
We are asked to find $A'(t)$ at that instant.
- The identity relating the functions in this case is $A = \pi r^2$.
- D_t of the identity is $A'(t) = 2\pi r r'(t)$.
- At the instant in question, $A'(t) = 2\pi \cdot 30 \cdot 4 = 240\pi \frac{cm^2}{min}$.

2.9 Differentials: Let $y = f(x)$ be a differentiable function of the independent variable x .

- Δx : an arbitrary increment in x
- dx : the differential $dx = \Delta x$.
- Δy : the exact change in y as the input variable changes from x to $x + \Delta x$, i.e.

$$\Delta y = f(x + \Delta x) - f(x).$$

- dy : is defined by $dy = f'(x) dx$. Geometrically dy is the tangent line approximation to the exact change Δy .



Example: The side lengths of a square are measured as $11 \text{ cm} \pm 0.2 \text{ cm}$. Estimate the area of the square, together with an estimate for the error.

Solution: The area of a square of side length x is given by $A = x^2$. So if our measurements were exact, the area of our square would be $11^2 = 121 \text{ cm}^2$. We can use differentials to estimate the error:

$$dA = A'(x)dx = 2x dx = 2 \cdot 11 \cdot (\pm .2) = \pm 4.4 \text{ cm}^2.$$

So we estimate that the area is

$$121 \pm 4.4 \text{ cm}^2.$$

We can also talk about the absolute error $|dA| = 4.4 \text{ cm}^2$, and the relative error

$$\frac{|dA|}{A} = \frac{4.4}{121} = 0.0364 = 3.64 \% .$$

3.1 Maxima and minima:

Definitions Let S be the domain of a function f (typically S is an interval).

Let c be a point in S . Then

- (i) $f(c)$ is a maximum value of f on S means that $f(c) \geq f(x)$ for all x in S .
- (ii) $f(c)$ is a minimum value of f on S means that $f(c) \leq f(x)$ for all x in S .
- (iii) $f(c)$ is an extreme value of f on S means that $f(c)$ is either a maximum value or a minimum value of f on S .

Theorem A) If f is continuous on a finite closed interval $[a, b]$ then f attains a maximum and minimum value (at least once) on its domain.

Theorem B) Let f be defined on an interval I . If $f(c)$ is an extreme value then c is one of the three sorts of critical points

- (i) c is an endpoint of the interval I
- (ii) $f'(c)$ exists and equals 0 (In this case c is called a "stationary point" - (name comes from physics.)
- (iii) $f'(c)$ does not exist. (In this case c is called a "singular point.")

3.2 Monotonicity and concavity

Definitions Let f be defined on an interval I (open, closed, or neither). Then

- (i) f is increasing on I means that for all points $x_2 > x_1$ in I , $f(x_2) > f(x_1)$.
- (ii) f is decreasing on I means that for all points $x_2 > x_1$ in I , $f(x_2) < f(x_1)$.
- (iii) f is strictly monotone on I means f is either increasing or decreasing on I .

Theorem A (Monotonicity Theorem) Let f be continuous on the interval I and differentiable at every interior point (i.e. non endpoint) of I . Then

- (i) If $f'(x) > 0$ for all x interior to I then f is increasing on I .
- (ii) If $f'(x) < 0$ for all x interior to I then f is decreasing on I .

Definitions Let f be differentiable on an open interval I . Then

- (i) f (or the graph of f) is concave up means that the slope function f' is increasing on I .
- (ii) f (or the graph of f) is concave down means that the slope function f' is decreasing on I .

Theorem B (Concavity Theorem) Let f be twice differentiable on the open interval I . Then

- (i) If $f''(x) > 0$ for all x in I , then f is concave up on I .
- (ii) If $f''(x) < 0$ for all x in I , then f is concave down on I .

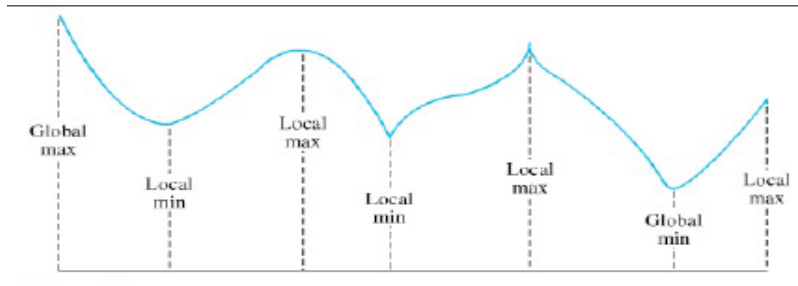
3.3 Local extrema

Definitions: Let f have domain S , with c in S .

(i) $f(c)$ is a local maximum value of f if there is an open interval (a, b) with $a < c < b$ so that $f(c)$ is the maximum value of f on the intersection of (a, b) with S

(ii) $f(c)$ is a local minimum value of f if there is an open interval (a, b) with $a < c < b$ so that $f(c)$ is the minimum value of f on the intersection of (a, b) with S .

(iii) $f(c)$ is a local extreme value of f if $f(c)$ is a local minimum value or a local maximum value of f .

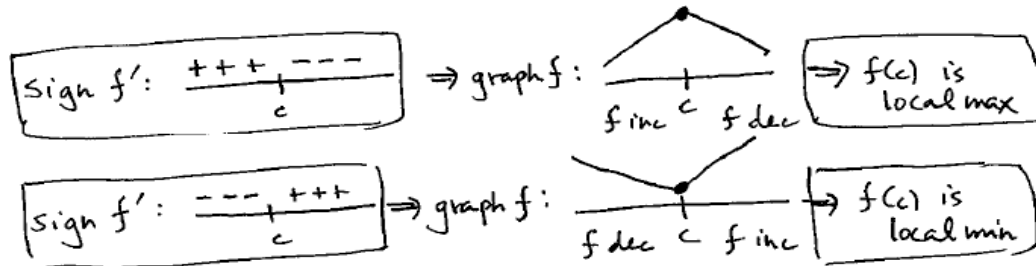


First derivative test: Let $f(x)$ be continuous in an open interval $I = (a, b)$ containing c , and differentiable in that interval except maybe at c . Then

(i) If $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$ then $f(c)$ is a local minimum.

(ii) If $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$ then $f(c)$ is a local maximum.

Pictorially,

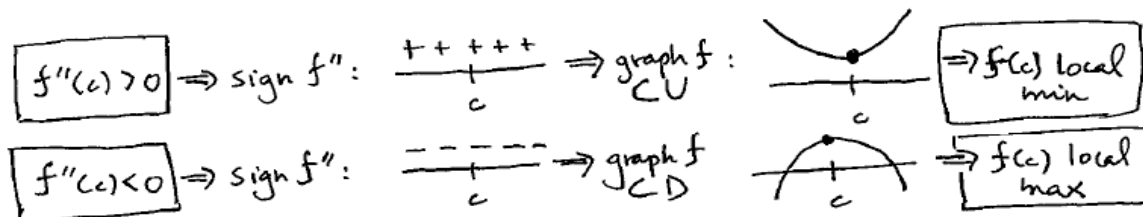


Second derivative test: Let $f(x)$ be differentiable on an open interval I containing c and let c be a stationary point ($f'(c) = 0$). If f'' is continuous on I and

(i) $f''(c) > 0$ then $f(c)$ is a local minimum.

(ii) $f''(c) < 0$ then $f(c)$ is a local maximum.

Pictorially,



Graphing: Use first and second derivative information to graph functions. This diagram is from section 3.5, but encapsulates what we do even though the example is somewhat more involved than what I would ask for a graphing problem on Exam 2.

