Math 1210-001
Tuesday Feb 9
WEB L110

2.5: Chain rule

The chain rule tells us how to differentiate compositions of functions. In other words, it tells us how to compute the rate of change of a composition two functions, in terms of the individual rates of change for each function. The chain rule makes intuitive sense when you think of real-world examples of rates of change, and keep track of units, so we'll start with two of those. Then we'll write the chain rule mathematically, see the mathematical reason it's true in general, and begin practicing it. You'll need to do a lot of practice outside of class as well, in order to develop proficiency in computations.

Exercise 1) Suppose you are driving south at 75 $\frac{miles}{hour}$, and the outside temperature is increasing at a rate of 0.04 $\frac{degrees}{mile}$ in the southerly direction. How fast is the outside temperature you're experiencing changing?

In one hour, travel 75 miles.
so temp goes up by $(75 \text{ miles})\left(.04 \frac{\text{degrees}}{\text{mile}}\right) = 3°$.

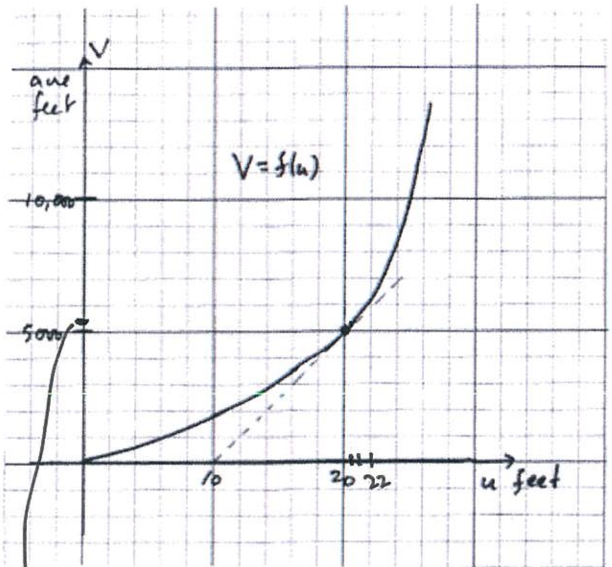
thus temp is increasing outside the moving car at a rate of $(75)(.04) = 3°/\text{hour}$.

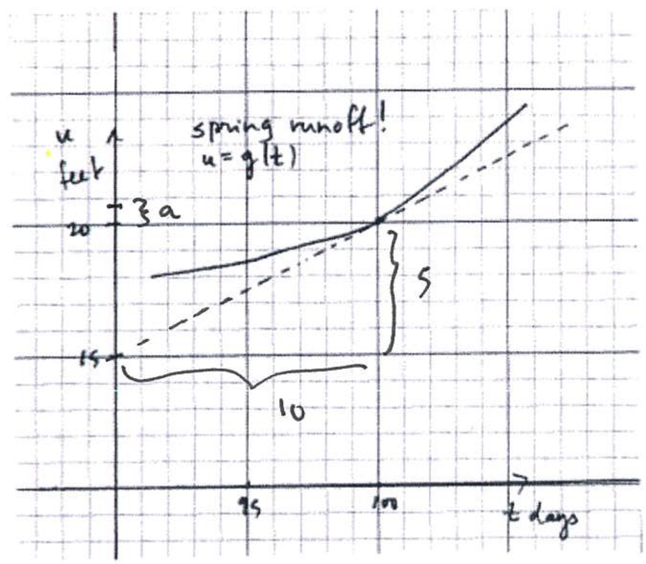
Remark Notice in the example above, that there is an underlying composition of two functions: Your location south is a function, say $g(t)$ miles south (of e.g. Salt Lake City) at time t hours, with $g'(t) = 75$ $\frac{miles}{hour}$. And there is a temperature function $T(y)$ *degrees* at location y miles south, with $T'(y) = .04 \frac{degrees}{mile}$. You were asked to find $D_t(T(g(t)))$.

Exercise 2 We consider a small town's reservoir and spring run-off. The volume V of the town's reservoir (in acre-feet) depends on the water depth u feet, measured from the deepest point in the reservoir, i.e. $V=f(u)$. The depth of the reservoir at day t of the calendar year is a function of t, $u=g(t)$. Suppose that at $t=100$ (March something) we have

$$t = 100 \ \ day$$
$$u = 20 \ \ ft$$
$$V = 5000 \ \ acre - ft.$$

Consult the two graphs below to answer questions a,b,c. Use secant approximations for a,b and tangent approximations for c.





a) How much does the water depth increase between $t = 100, 101$?

about .7 feet.

approx 400 acre ft/day

b) About how much does the reservoir volume increase between $t = 100, 101$?

from graph at left (it's hard to be precise), maybe by 400 acre feet (or maybe 300?)

c) About how fast is the reservoir volume changing at $t = 100$. (Use tangent line approximations. Keep track of units.)

using tangent lines at $t=100$ inst. rate of change of depth is $\approx .5$ ft/day

and from graph at left, when $u=20'$ volume is increasing at rate of 500 $\dfrac{\text{acreft of vol}}{\text{ft of depth}}$.

so inst. rate is $(500)(.5)=250$ acreft/day.

The moral of Exercises 1,2, is that when functions are composed, the rate of change of the composition is the product of the rates of change for each function being composed. This is what the chain rule says. Let's work our way to the precise statement and its proof using the Δ ("change in") notation.

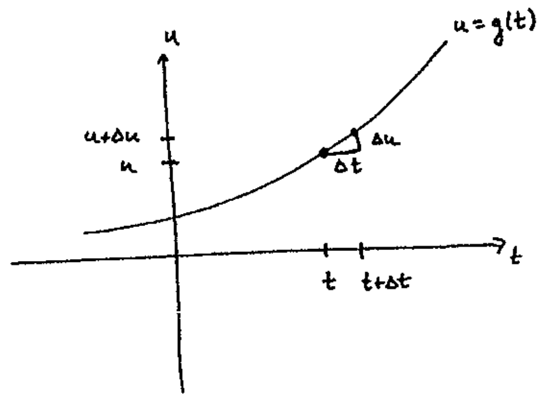
For $u = g(t)$ we write "Δt" for "change in t", instead of "h".

and we write "Δu" for the corresponding change in u as t increments to $t + \Delta t$,

$$\Delta u = g(t + \Delta t) - g(t)\ .$$

So, converting notations,

$$D_t\, g(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = \lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t}$$



Use these notation conventions, and consider the general composition

$$u = g(t) \qquad\qquad g'(t) = \lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t}$$

$$V = f(u) \qquad\qquad f'(u) = \lim_{\Delta u \to 0} \frac{\Delta V}{\Delta u}$$

Then

$$\begin{aligned} D_t f(g(t)) &= \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta u} \cdot \frac{\Delta u}{\Delta t} \quad \text{(assuming } \Delta u \neq 0) \\ &= f'(u) \cdot g'(t) \end{aligned}$$

Chain Rule!

$$D_t\big(f(g(t)) = f'(g(t)) \cdot g'(t)\ .$$

"The derivative of the composition function is the derivative of the outside function evaluated at the inside function, times the derivative of the inside function." OR "The rate of change of the composition of two functions is the product of their rates of change," evaluated at the appropriate input values.

Exercise 3) Write each composition as $f \circ g$, and use the chain rule to find the derivative of the composition function.

3a) $F(x) = (x^3 + x)^2$ (check answer with "foil")

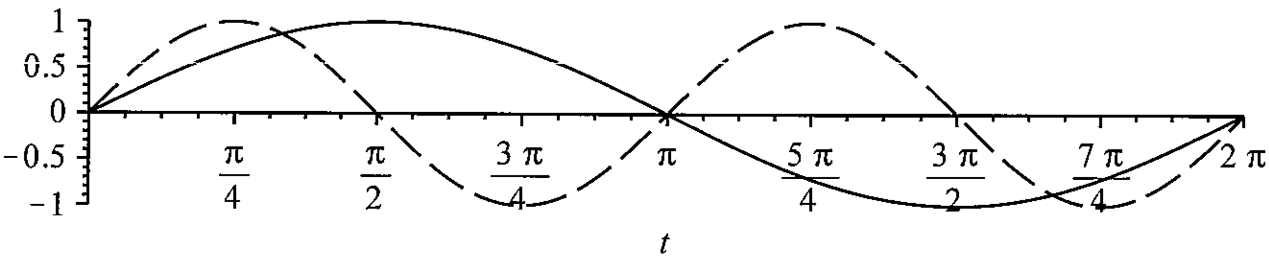
$D_x f(g(x)) = f'(g(x)) \cdot g'(x)$ $= 2(x^3+x)^1(3x^2+1)$.

$g(x) = x^3 + x$ $\quad g'(x) = 3x^2 + 1$

$f(u) = u^2$ $\quad f'(u) = 2u$

3b) $F(t) = \sin(2t)$ (understand your answer geometrically by looking at the graph below, from Monday's notes).

(see Feb 10 notes)



3c) $G(r) = (\tan(r))^4$

$= f(g(r))$

$g(r) = \tan(r)$ $\quad g'(r) = \sec^2(r)$

$f(u) = u^4$ $\quad f'(u) = 4u^3$

$G'(r) = f'(g(r))\, g'(r)$

$= 4(\tan(r))^3 \cdot \sec^2 r$

3d) $H(t) = [\cos(3t^3)]^2$. $\quad H'(t) = 2[\cos(3t^3)]^1 \underbrace{D_t \cos(3t^3)}_{-\sin(3t^3))\cdot 9t^2}$

$g(t) = 3t^3$ $\quad g'(t) = 9t^2$

$h(u) = \cos u$ $\quad h'(u) = -\sin u$

$f(v) = v^2$ $\quad f'(v) = 2v$

$\boxed{H'(t) = 2\cos(3t^3)(-\sin(3t^3))\cdot 9t^2}$