

Math 1210-001  
Tuesday Feb 16  
WEB L110

## 2.7 Implicit differentiation.

Sections 2.7 and 2.8 are important applications of the chain rule. In section 2.7 we use identity equations and the chain rule to find the derivatives of functions at given points on their graphs, without needing to know the explicit formulas for the functions. This method is called implicit differentiation. In section 2.8 we use identities involving several different functions (of time) at once, along with the chain rule, in order to figure out how fast one of those functions is changing with respect to time, in terms of how fast the others are. This topic is called related rates.

Exercise 1 (warm-up, and related to lab 4, problem 1b).

1a) Use the limit definition of derivative to show the power rule with  $n = \frac{1}{2}$ , namely

$$D_x \left( x^{\frac{1}{2}} \right) = \frac{1}{2} x^{-\frac{1}{2}}$$

1b) Let  $g(x) = \sqrt{x}$ , for  $x > 0$ . Then we have the identity

$$g(x)^2 = x \quad \text{for all } x > 0.$$

Thus

$$D_x g(x)^2 = D_x x.$$

Use the chain rule to expand the equality above, solve for  $g'(x)$  and deduce the power rule with  $n = \frac{1}{2}$ .

1c) Now let's check the power rule for *every* rational number power, using the last method. Note that our bag of algebraic tricks would *not* work for the limit definition method in this generality. Let

$$y = g(x) = x^n = x^{\frac{p}{q}}$$

where  $p, q$  are any two non-zero integers, and where  $x > 0$  if  $q$  is an even integer. Then, taking  $q^{th}$  powers of both sides of the equation, we have the identity

$$y^q = x^p$$

for  $y = g(x)$ . Differentiate both sides of this identity - using the chain rule and the power rule for integers - to deduce the power rule,  $D_x x^n = n x^{n-1}$  for all rational number powers.

Exercise 2) Find

$$D_x \left( \sqrt[3]{\cos(2x)} \right).$$

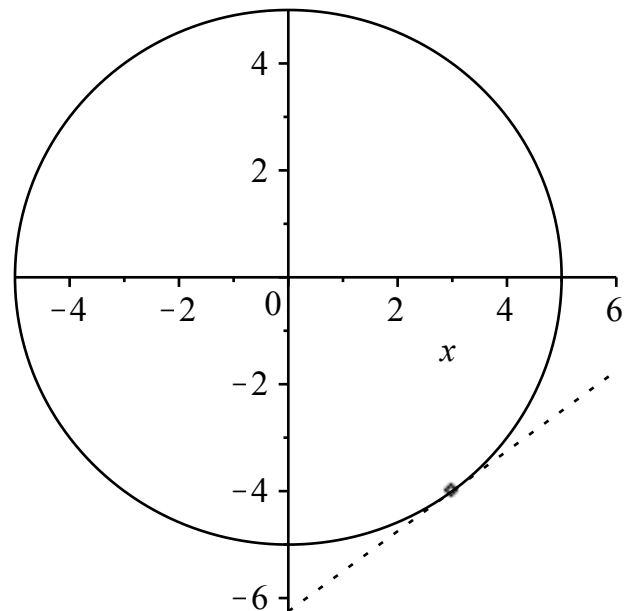
Exercise 3) The graph of the equation  $x^2 + y^2 = 25$  is a circle of radius 5, centered at the origin. We will find the slope of the graph of this equation, at the point  $(3, -4)$ , two ways:

3a) Explicitly solve for the function  $y = g(x)$  whose graph contains the point  $(3, -4)$  and is part of the graph of the circle. Compute  $g'(3)$  to find the slope.

3b) The function  $y = g(x)$  in part (a) satisfies the identity  
$$x^2 + y^2 = 25.$$

Use this implicit definition of  $g(x)$ , differentiate both sides of the identity above with respect to  $x$ , and find the slope of the tangent line at the point  $(3, -4)$  that way.

3c) What's the equation of the tangent line to this circle, at the point  $(3, -4)$ ?



A lot of times the equation that implicitly defines  $y$  as a function of  $x$ , near a given point on the graph of the equation, is not easily explicitly solvable for  $y$  as a function of  $x$ . (In other words, the previous example was an exception, not the rule.) However, implicit differentiation will always work, provided the graph of the equation actually has a non-vertical tangent line at the given point.

Exercise 4) Find the equation of the tangent line to the graph of

$$2 \cdot x^3 + y^2 + x^4 \cdot y^4 = 4$$

at the point  $(1, 1)$  on the graph.

