Math 1210-001
Monday Apr 4
WEB L112

4.2-4.4 continued

Today:

- Reinterpret the "area" problem that we've been using as our visualization for the definite integral as a physics problem instead where $f(t)$ is a velocity function, and realize how Newton probably came up with the Fundamental Theorem of Calculus shortcut for computing definite integrals. This is in Friday's notes:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

- Check the two area examples with the FTC, at the end of Friday's notes.

- There are actually two parts to the Fundamental Theorem of Calculus - our text calls them the Part I of the FTC (section 4.3) and Part II (section 4.4). It's actually Part II that is stated above. But there's a subtlety that is swept under the rug in the Friday notes explanation - namely we assume there that the function $f(t)$ actually has an antiderivative. (In the Friday notes explanation, we assume that $f(t)$ is a velocity function, so we're implicitly assuming there is a background position function $s(t)$ with $s'(t) = f(t)$. This certainly seems reasonable. :-)) Part I of the FTC exhibits an explicit formula for an antiderivative function, so takes care of this subtlety. It is also used in Calc 2, to define an antiderivative for the function $f(x) = \frac{1}{x}$. Similar formulas to the "accumulation function" antiderivative below also arise in statistics.

<u>The Fundamental Theorem of Calculus</u>: Let $f(x)$ be continuous on $[a, b]$. Define the definite integral on $[a, b]$ and subintervals using limits of Riemann sums. Define the "accumulation function" $\mathcal{A}(x)$ by integrating f from a to x:

$$\mathcal{A}(x) = \int_a^x f(t)\,\mathrm{d}t$$

<u>Part I</u>: $\mathcal{A}(x)$ is an antiderivative of $f(x)$, i.e.

$$D_x\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x).$$

(So all antiderivatives are of the form $\mathcal{A}(x) + C$.

<u>Part II</u>: Let $F(x)$ be any antiderivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

We'll study in detail why <u>Part I</u> is true, tomorrow. Today we'll continue to work examples of Part II, and also some that relate to Part I. There is lots of further practice in your WebWork assignment.

Exercise 1) Compute

$$\int_{-1}^{3} 5\,x^2 - 3\,x + 1 \, \mathrm{d}x$$

Exercise 2) Compute

$$\int_{0}^{2} x\,\sqrt{x^2 + 5} \, \mathrm{d}x$$

Exercise 3) Compute

$$D_x\left(\int_{0}^{x} 1 + 2\,t \, \mathrm{d}t\right)$$

two ways:
(i) by first computing the definite integral in terms of x, and then taking its x-derivative
(ii) Apply Part I of the FTC to do the computation much more quickly.

Exercise 4) Compute

$$D_x\left(\int_{x^2}^{2\,x} t^2 \, \mathrm{d}t\right)$$

two ways:
(i) by first computing the definite integral in terms of x and then taking its x-derivative.

(ii) Using this huge shortcut (which may be all you have available on some WebWork problems):

$$\int_{g(x)}^{h(x)} f(t)\ \mathrm{d}t = F(h(x)) - F(g(x))$$

where F is an antiderivative of f. Thus

$$\begin{aligned} \mathrm{D}_x \int_{g(x)}^{h(x)} f(t)\ \mathrm{d}t &= \mathrm{D}_x(F(h(x)) - F(g(x))) \\ &= F'(h(x))h'(x) - F'(g(x))g'(x) \\ &= f(h(x))h'(x) - f(g(x))g'(x). \end{aligned}$$

So you can do this sort of problem without any anti-differentiation at all.

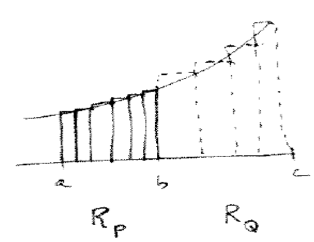
Properties of definite integrals (we may not have time to discuss these until Tuesday):

- Interval additivity: If $a < b < c$ then

$$\int_a^c f(x)\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x + \int_b^c f(x)\ \mathrm{d}x.$$

because of R_P is a Riemann sum for $[a, b]$ and
R_Q is a Riemann sum for $[b, c]$ then $R_P + R_Q$ is a Riemann sum for $[a, c]$ and

$$\begin{aligned} \int_a^c f(x)\ \mathrm{d}x &= \lim_{\|P\|,\ \|Q\| \to 0} (R_P + R_Q) \\ &= \lim_{\|P\| \to 0} R_P + \lim_{\|Q\| \to 0} R_Q \\ &= \int_a^b f(x)\ \mathrm{d}x + \int_b^c f(x)\ \mathrm{d}x\ . \end{aligned}$$



- Linearity with respect to the integrand:

$$\int_a^b c f(x)\ \mathrm{d}x = c \int_a^b f(x)\ \mathrm{d}x$$

$$\int_a^b f(x) + g(x)\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x + \int_a^b g(x)\ \mathrm{d}x.$$

One way to see this is because antidifferentiation also satisfies these properties. Another way is directly from the Riemann sum definitions:

$$\sum_{i=1}^{n} c f(\underline{x}_i)\Delta x_i = c \sum_{i=1}^{n} f(\underline{x}_i)\Delta x_i$$

$$\sum_{i=1}^{n} \left(f(\underline{x}_i) + g(\underline{x}_i)\right)\Delta x_i = \sum_{i=1}^{n} f(\underline{x}_i)\Delta x_i + \sum_{i=1}^{n} g(\underline{x}_i)\Delta x_i$$

Take the limit of these Riemann sum expressions as the maximum sub-interval width ($\|P\|$) goes to zero to get the definite integral identities.

- inequality estimates: If $f(x) \leq g(x)$ on $[a, b]$ then

$$\int_a^b f(x)\ \mathrm{d}x \leq \int_a^b g(x)\ \mathrm{d}x.$$

This is because

$$\sum_{i=1}^{n} f(\underline{x}_i)\Delta x_i \leq \sum_{i=1}^{n} g(\underline{x}_i)\Delta x_i.$$

Then take the $\lim_{\|P\| \to 0}$ to get the definite integral inequality.