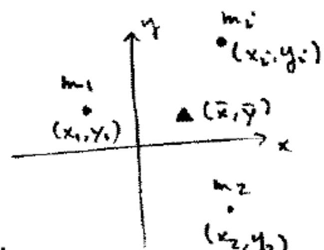
Math 1210-001
Monday Apr 25
WEB L112

- Webwork or lab question clarifications?
- Finish Friday's notes on surface area and on mass/moments/centers of mass for 1-dimensional objects.

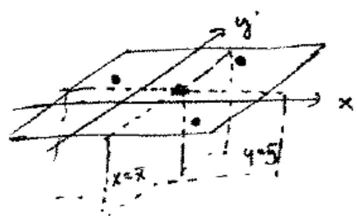
We may be able to begin the discussion below today, from the second half of section 5.6. If not, we can do it tomorrow.

two-dimensional version for discussion of mass, moments, center of mass:

Consider a collection of point masses in the $x - y$ plane. This of the masses attached at points on a mass-less horizontal plane. If the points balance on a vertical plane along $x = \underline{x}$, and on a vertical plane along $y = \underline{y}$, then they will balance at the point $(\underline{x}, \underline{y})$...the "center of mass".



3-d perspective



$$M_x = \text{ moment with respect to the } x - axis := \sum_{i=1}^{n} m_i y_i$$

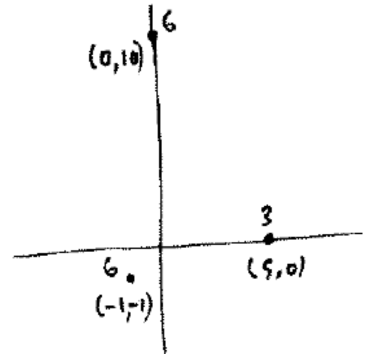
$$M_y = \text{ moment with respect to the } y - axis := \sum_{i=1}^{n} m_i x_i \quad \text{(looks like earlier 1-d formula)}$$

$$m := \sum_{i=1}^{n} m_i \quad \text{total mass}$$

$$\underline{x} = \frac{M_y}{m} = \frac{\left(\sum_{i=1}^{n} m_i x_i\right)}{m} \quad \text{(just like old formula)}$$

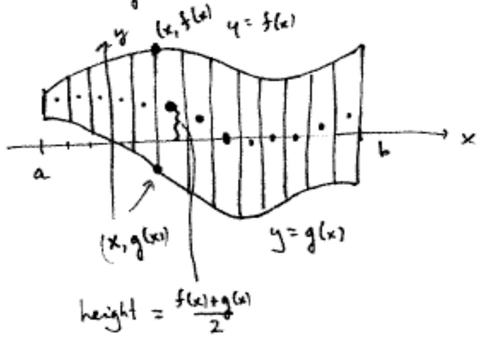
$$\underline{y} = \frac{M_x}{m} = \frac{\left(\sum_{i=1}^{n} m_i y_i\right)}{m}.$$

Exercise 1) Find the center of mass $(\underline{x}, \underline{y})$ if $m_1 = 3$ is at $(5, 0)$; $m_2 = 1$ is at $(0, 10)$; $m_3 = 6$ is at $(-1, -1)$:



In multivariable calculus (Math 2210) you learn how to find moments and centers of mass for 2-d and 3-d objects with varying mass-density functions. Using single variable calculus we can handle lamina, (thin planar sheets), with constant density $\delta \dfrac{mass}{area}$:

Consider a region:



this one's tricky!

ht of ctr of mass of vertical strip

$$\Delta m \approx \delta\,(f(x) - g(x))\,\Delta x$$

$$\Delta M_y \approx x(\Delta m) \approx x\,\delta\,(f(x) - g(x))\,\Delta x$$

$$\Delta M_x \approx \frac{(f(x)+g(x))}{2}\,\Delta m = \frac{f(x)+g(x)}{2}\,\delta\,(f(x)-g(x))\,\Delta x = \frac{\delta}{2}\left(f^2(x) - g^2(x)\right)\Delta x$$

$$m = \int_a^b \delta\,(f(x) - g(x))\,dx, \quad M_y = \int_a^b \delta\, x\,(f(x) - g(x))\,dx,$$

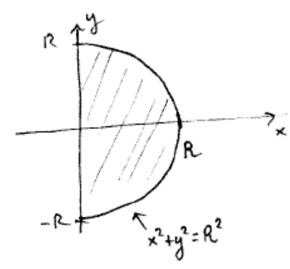
$$M_x = \int_a^b \frac{\delta}{2}\left(f^2(x) - g^2(x)\right)dx$$

for $\bar{x}$ & $\bar{y}$ the δ's cancel out:

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x\,(f(x) - g(x))\,dx}{\int_a^b f(x) - g(x)\,dx} \leftarrow \text{area! } A$$

$$\bar{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}\left(f^2(x) - g^2(x)\right)dx}{A}$$

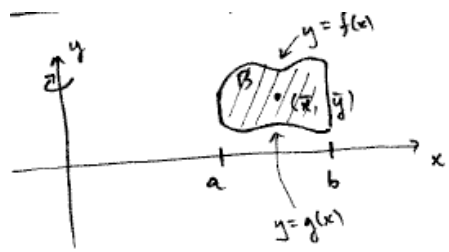
Exercise 2) Find the center of mass of a half-disk lamina. For these constant mass-density lamina we often call the center of mass $(\underline{x}, \underline{y})$ the centroid.



Amazing fact ("Pappus' Theorem, circa A.D. 300 - those amazing Greeks!) Consider a volume of revolution obtained by rotating a region B about an axis on one side of it. Let (x, y) be the centroid of B. Let ρ be the distance from the axis to the centroid. Let B have area A. Then the volume of revolution is

$$V = (2\pi\rho)A.$$

This theorem follows directly from the method of cylindrical shells in the case that one rotates a region bounded above and below by graphs of functions, about the $y-axis$. You might want to try checking it.



Example: Predict the answer you will get to your last lab problem #9. In that problem the disk of radius 1, centered at $(2, 0)$ is rotated about the $y-axis$. The centroid of the disk is its center, by symmetry.