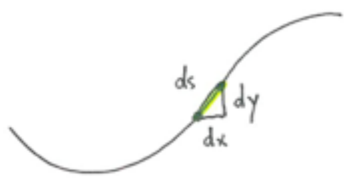
Math 1210-001
Friday Apr 22
WEB L112

- Webwork or lab question clarifications?
- Today we'll finish section 5.4 on curve length and surface area for surfaces of revolution. We may have a chance to begin section 5.6 on mass, moments, and centers of mass.

Summary of Wednesday discussion for length of curves:



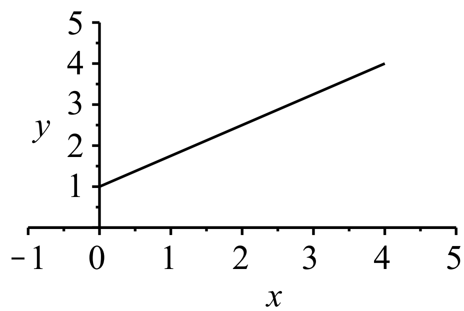
$$L = \int ds = \int \sqrt{(dx)^2+(dy)^2}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \quad \ldots \text{if } \begin{array}{l} x=f(t) \\ y=g(t) \\ a\le t\le b \end{array}$$

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \ldots \text{if } \begin{array}{l} y=f(x) \\ a\le x\le b \end{array}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2+1} \qquad \text{if } \begin{array}{l} x=g(y) \\ a\le y\le b \end{array}$$

This exercise was #3 on Wednesday's notes:
Exercise 1) Consider the line segment from $(0, 1)$ to $(4, 4)$.

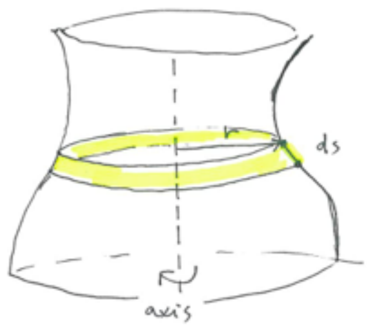


1a) Find the length using the Pythagorean Theorem.

1b) Express the line as the graph $y = f(x)$ and find the length with one of the integrals above.

1c) Express the line as a graph $x = g(y)$ and find the length using one of the integrals above.

Surface area of revolution: Partition the curve; rotate an arc of length ds get a ribbon of area $\Delta A \approx (2\,\pi\,\underline{r})\Delta s$, where $\underline{r}$ is the average distance to the axis. (This approximating is exact if the arc is a line segment and the ribbon is a piece of a cone, see #31 page 301.) Add up the areas in a Riemann sum, take the limit, and you get surface area integral:



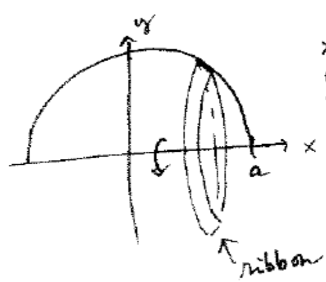
$\Delta A \approx 2\pi\,\bar{r}\cdot(\Delta s)$ $\quad \bar{r}$ = average radius

$dA = 2\pi r\, ds$

so if $x = f(t)$
$y = g(t)$
$a \le t \le b$

$$A = \int \underset{\substack{\| \\ 2\pi r\, ds}}{dA} = \int_a^b 2\pi\, r(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt$$

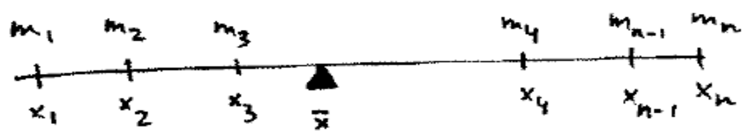
Exercise 2) Verify the formula for the surface area of a sphere of radius a, obtained by rotating the upper half circle about the $x-axis$. Use the parameterization below. (In your Lab problem #5 you are asked to do this using the graph formula $y = \sqrt{a^2 - x^2}$ instead.)



$x = a\cos t$
$y = a\sin t$ $\qquad 0 \le t \le \pi$

5.6 Moments and center of mass

Picture a very crowded teeter-tottler:



The balance point for the masses distributed along a line is the $\underline{x}$ which yields a zero "moment":

$$\sum_{i=1}^{n} \left(x_i - \underline{x}\right) m_i = 0.$$

(This is the point $\underline{x}$ at which it takes no net work to rotate the configuration a tiny amount, e.g. working against gravity.) Expanding the summation above lets us solve for the "center of mass" $\underline{x}$:

$$\sum_{i=1}^{n} x_i m_i - \underline{x} \sum_{i=1}^{n} m_i = 0$$

$$\underline{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} := \frac{M}{m}.$$

"M" is called the moment (with respect to the origin) of the mass configuration, and "m" is the total mass.

Exercise 3) What is the center of mass for the indicated masses located at the indicated points on the number line below?

Add calculus! Consider a rod with varying mass density (mass/length) $\delta(x)$, $a \le x \le b$. Partition, approximate, and take limits to get the analogous expressions as definite integrals:

$$\Delta m_i \approx \delta(x_i)\Delta x_i$$

$$\underline{x} \approx \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} x_i \delta(x_i)\Delta x_i}{\sum_{i=1}^{n} \delta(x_i)\Delta x_i}\,.$$

Take limit as maximum $\Delta x_i \rightarrow 0$:

$$M := \int_a^b x\,\delta(x)\,\mathrm{d}x \quad \text{moment with respect to } x = 0$$

$$m := \int_a^b \delta(x)\,\mathrm{d}x \quad \text{total mass}$$

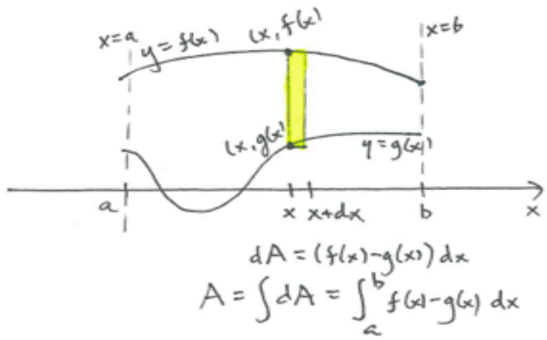
$$\underline{x} := \frac{M}{m} \quad \text{center of mass}$$

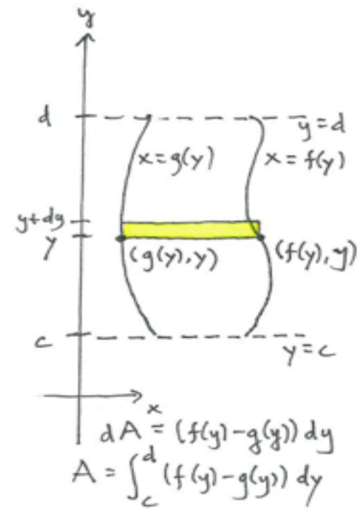
<u>Exercise 4)</u> (see Lab #7, Webwork #9) A 10 cm long wire has varying density $\delta(x) = .004\,x^3\ \frac{gm}{cm}$, $0 \le x \le 10$.

<u>4a)</u> Find its total mass

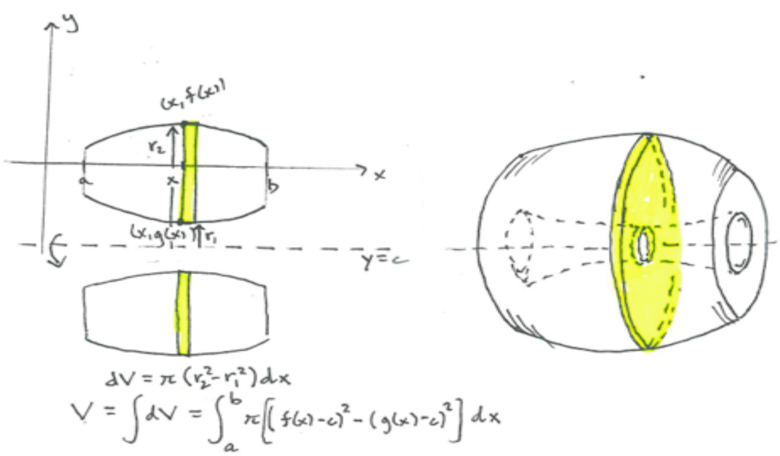
<u>4b)</u> Find its center of mass.

Areas (5.1):





Volumes by disks and washers (5.2), slice perpendicular to axis of rotation:



Volumes by cylindrical shells (5.3), slice parallel to axis of rotation: