Math 1210-001
Monday Apr 11
WEB L112

4.5: The Mean Value Theorem for integrals; integral shortcuts using symmetry.

We will start Chapter 5 tomorrow - it contains a number of useful applications for definite integrals. Section 4.5 (and a piece of 4.4) contain some useful shortcuts. In 4.5 there is also a discussion of what "the average value of f on the interval $[a, b]$" means, and a related theorem.

Mean value theorem for integrals.

Definition: If f is integrable over the interval $[a, b]$ then the average value of f on $[a, b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\, dx.$$

Reason for the definition: Partition $[a, b]$ into n equal subintervals as we're used to, with widths $\Delta x = \dfrac{b-a}{n}$:

The standard average of the values $f(x_1), f(x_2), \ldots f(x_n)$ is their sum, divided by n:

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i)\ .$$

This turns out to be closely related to the corresponding Riemann sum for $\int_a^b f(x)\, dx$ with right endpoints:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \Delta x \sum_{i=1}^{n} f(x_i)$$

$$R_n = \frac{(b-a)}{n}\sum_{i=1}^{n} f(x_i).$$

Comparing, we see

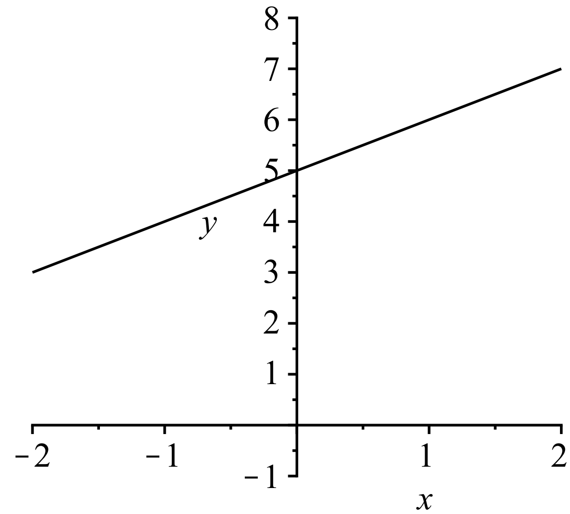
$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a} R_n$$

So,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

provided the limit of the Riemann sums exists (which is what we mean by f being integrable on $[a, b]$.) As we've discussed, this is always the case if f is continuous on $[a, b]$.

Exercise 1) Find the average value of $f(x) = x + 5$ on the interval $[-2, 2]$. Compare to the geometric picture of the graph. Notice that it is always the case, as illustrated here, that a constant function defined to be the average value of f has the same integral as f, over the relevant interval $[a, b]$.



Theorem (Mean value theorem for integrals): Let f be continuous on $[a, b]$. Then there is at least on c, $a < c < b$, with

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

proof: This is actually an application of the Mean Value Theorem for derivatives and part I of the Fundamental Theorem of Calculus, applied to the accumulation function

$$\mathcal{A}(x) = \int_a^x f(t)\,\mathrm{d}t.$$

In fact, there is a c so that

$$\frac{\mathcal{A}(b) - \mathcal{A}(a)}{b-a} = \mathcal{A}'(c) = f(c).$$

Since $\mathcal{A}(b) = \int_a^b f(x)\,\mathrm{d}x$ and $\mathcal{A}(a) = \int_a^a f(x)\,\mathrm{d}x = 0$ the expression

$$\frac{\mathcal{A}(b) - \mathcal{A}(a)}{b-a} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

is just the average value.

□

Definite integral shortcuts:

1) Shortcut in using substitution for definite integrals:
We seek to evaluate

$$\int_a^b f(g(x))g'(x)\ \mathrm{d}x$$

Method 1 (not the shortcut - this is how we've been doing them up to now) use u-substitution, $u = g(x)$, $du = g'(x)dx$, so the indefinite integral

$$\int f(g(x))\ \mathrm{d}x = \int f(u)\ \mathrm{d}u = F(u) + C = F(g(x)) + C.$$

Then plug in the x-limits:

$$\int_a^b f(g(x))g'(x)\ \mathrm{d}x = F(g(b)) - F(g(a)).$$

Method 2 (shortcut): change the limits (interval endpoints) to $u - limits$ at the same time you make the u-substitution. It yields the same answer as above, so we can use the shortcut:

$$\int_a^b f(g(x))g'(x)\ \mathrm{d}x$$

Use the u-substitution $u = g(x)$, $du = g'(x)dx$. Also, when $x = a$, $u = g(a)$; when $x = b$, $u = g(b)$.
Substitute these endpoints too:

$$\int_a^b f(g(x))g'(x)\ \mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\ \mathrm{d}u = F(g(b)) - F(g(a)).$$

Exercise 1) Compute

$$\int_0^3 x\left(x^2 + 1\right)^9\ \mathrm{d}x$$

both ways to compare.

Symmetry shortcuts in definite integrals:

recall:

Definitions

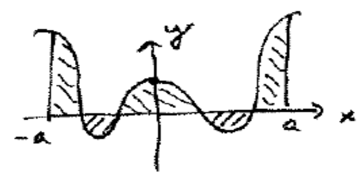
(1) $f(x)$ is an even function means $f(-x) = f(x)$ holds for all x. (For example, polynomials $p(x)$ in which only even powers of x appear are even functions.) In terms of the graph of f, for an even function whenever (x, y) is on the graph of f then so is $(-x, y)$. In other words, the graph is symmetric with respect to the x-axis.

(2) $f(x)$ is an odd function means $f(-x) = -f(x)$ holds for all x. (For example, polynomials $p(x)$ in which only odd powers of x appear are odd functions.) In terms of the graph of f, for an odd function whenever (x, y) is on the graph of f, then so is $(-x, -y)$. We call such graphs symmetric with respect to the origin.

Theorem)

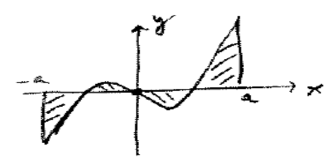
(1) If $f(x)$ is an even function then

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = 2\int_{0}^{a} f(x)\ \mathrm{d}x.$$



(2) If $f(x)$ is an odd function then

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = 0.$$



Both parts of this Theorem makes geometric sense, in terms of signed area either adding up to double, or canceling out. To give an algebraic proof, write

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = \int_{-a}^{0} f(x)\ \mathrm{d}x + \int_{0}^{a} f(x)\ \mathrm{d}x.$$

Do a u-substitution in the first integral, and then use the definition that

$$\int_{c}^{d} f(x)\ \mathrm{d}x = -\int_{d}^{c} f(x)\ \mathrm{d}x$$

that we talked about last week.

Exercise 2) Check the following properties of products of odd and even functions:

1) $even \cdot even = even$
2) $even \cdot odd = odd$!
3) $odd \cdot odd = even$!

Exercise 3) a) Use symmetry to save steps in computing

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 + x^2 \sin(x) - 2\,x \cos(x) + 4 \cos(x)\, dx.$$

b) What is the average value of the function above, on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?