

Math 1210-001
 Friday Apr 1
 WEB L112

Today:

- Look over the Wednesday notes together, on the general definition of Riemann integral (section 4.2 material).
- Do one example carefully today, using the template Dapeng discussed in lab.
- Reinterpret the "area" problem in that example as a physics problem instead, and realize how Newton probably came up with the Fundamental Theorem of Calculus shortcut for computing definite integrals.
- Check a few area/physics examples with the FTC.

Today's example: Compute the definite integral

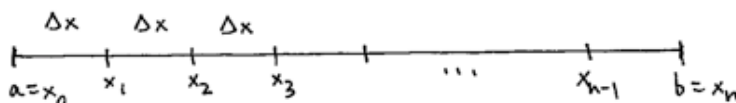
$$\int_{-1}^3 2x + 3 \, dx$$

from the limit of Riemann sums definition, using right endpoints.

Template for equal subdivisions and right endpoints is this formula:

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x.$$

(With left endpoints the $f(x_i)$ would be replaced with $f(x_{i-1})$). The general template with n equal subdivisions expanded:



Since the total length of the interval $[a, b]$ is $b - a$, each subinterval has width

$$\Delta x = \frac{(b - a)}{n}.$$

The sub-interval endpoints are

$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x = a + \frac{(b - a)}{n} \\ x_2 &= a + 2 \Delta x = a + \frac{2(b - a)}{n} \\ x_i &= a + i \Delta x = a + i \frac{(b - a)}{n}. \end{aligned}$$

The first interval is

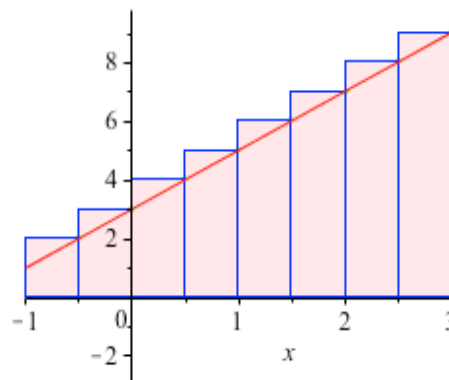
$$I_1 = [x_0, x_1]$$

and the i^{th} interval is

$$I_i = [x_{i-1}, x_i].$$

Picture when $n = 8$ subdivisions of length 0.5 each, for right sum Riemann sum approximation to

$$\int_{-1}^3 2x + 3 \, dx$$



$$f(x) =$$

$$[a, b] =$$

$$\Delta x =$$

$$x_i =$$

$$f(x_i) =$$

$$f(x_i) \Delta x =$$

$$\sum_{i=1}^n f(x_i) \Delta x =$$

$$\sum_{i=1}^n f(x_i) \Delta x \text{ in terms of } n \text{ only (use appropriate formula from next page):}$$

$$\int_{-1}^3 2x + 3 \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x =$$

Check: We know how to compute areas of trapezoids. Did we get the correct area?

The "magic formulas" for some special sums (see page 218):

$$\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{i=1}^n i^2 = 1 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{i=1}^n i^4 = 1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

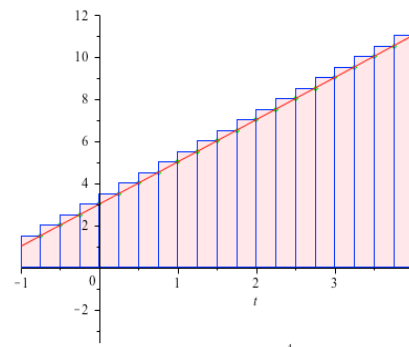
Reinterpretation of the previous example: Let

$$v(t) = 2t + 3$$

be the velocity of a turtle on a straight-line walk, in $\frac{\text{meters}}{\text{minute}}$. (So for our time interval, the turtle is walking in the positive direction.)

Then $v(t_i)\Delta t$ is an approximation to how far the turtle traveled between t_{i-1} and t_i (rate*time=distance), and a pretty good one when Δt is small, since the turtle's velocity won't be changing much on short subintervals. Notice that the units are

$$\frac{\text{meters}}{\text{minute}} \cdot \text{minutes} = \text{meters}.$$



So the Riemann sum

$$\sum_{i=1}^n v(t_i) \Delta t$$

approximates the net distance traveled during the time interval $[-1, 3]$ (more generally on any interval $[a, b]$ and for continuous velocity function $v(t)$.)

$$\int_a^b v(t) dt = \lim_{n \rightarrow \infty} \sum_{i=1}^n v(t_i) \Delta t$$

will be the actual net distance traveled. So if the position function is denoted by $s(t)$, the value of the integral will be the net distance $s(b) - s(a)$.

In our example, since the velocity is

$$v(t) = 2t + 3$$

The position function must be one of the velocity antiderivatives,

$$s(t) = t^2 + 3t + C$$

And the value of the integral must be

$$s(3) - s(-1) = (9 + 9 + C) - (1 - 3 + C) = 20. !!!$$

(Note that the choice of C doesn't matter as far as the final answer.) When computing

$$\int_a^b f(x) dx$$

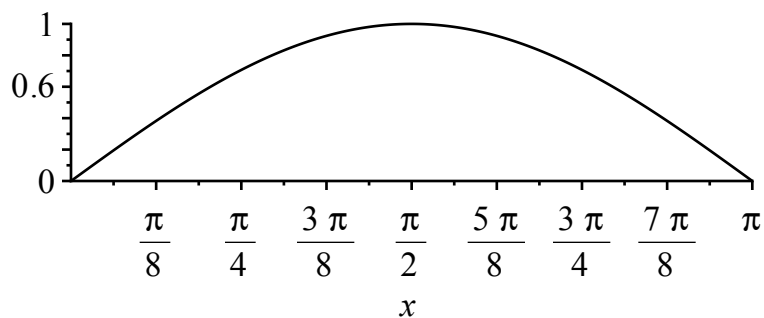
One could "pretend" that $f(x)$ is a velocity function and that x is time. One deduces that

$$\int_a^b f(x) dx = F(b) - F(a) \quad (\text{which we abbreviate as } F(x) \Big|_a^b)$$

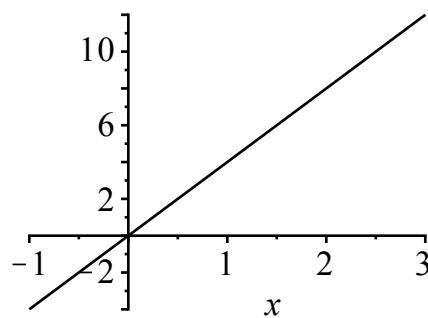
where $F(x)$ is any antiderivative of $f(x)$. This is The Fundamental Theorem of Calculus (part 2), section 4.4

So, e.g. we can compute areas really easily. For any function you can antidifferentiate, you can compute the (signed) area between it and the x -axis without using Riemann sums and limits:

1) $\int_0^{\pi} \sin(x) \, dx =$



2) $\int_{-1}^3 4x \, dx =$



(integrals give negative area when the graph is below the x - axis)