Math 1210-3
Monday March 24. Welcome back!!

§3.8 Antiderivatives

Recall, from early in the course and from repeated physics problems,

Definition: $F(x)$ is an antiderivative of $f(x)$ on the interval I
if and only if $D_x F(x) = f(x) \quad \forall x \in I$

Exercise 1a) Find an antiderivative for $f(x) = x^3$ on $(-\infty, \infty)$.

1b) Find all antideriv's for $f(x) = x^3$ on $(-\infty, \infty)$.

The correct answer to 1b) is justified by

Theorem A (the "+C" theorem)

a) If $F(x)$ is an antideriv. of $f(x)$ on interval I, and if C is a constant, then $F(x) + C$ is an antiderivative of $f(x)$ too.

b) If $F(x)$ is an antideriv. of $f(x)$ on I, then the only other antiderivs are $F(x) + C$, where C is a const.

Exercise 2 : Prove A.a)

<u>proof of A.b</u>: Another way to state A.b is that every other antideriv. of f differs from F by a constant, i.e. if $G(x)$ is another antideriv, then

$$\underbrace{G(x)-F(x)}_{\text{Call this expression } H(x)} = C \qquad \text{for some const. } C.$$

<u>Exercise 3</u> Why is $H'(x)=0 \;\; \forall x \in I$?

Since $H'(x)=0 \;\; \forall x \in I$, it sure seems like $H(x)$ must be constant.
Let's prove this fact with the mean value theorem:

Pick any $x_0 \in I$. Set $C = H(x_0)$.

$$\left.\begin{array}{l} b > x_0 \Rightarrow \dfrac{H(b)-H(x_0)}{b-x_0} = \underbrace{H'(c)}_{=0!} \quad \text{some } x_0 < c < b \\ \qquad\qquad \Rightarrow H(b) = \underbrace{H(x_0)}_{=C}, \quad \text{any } b \in I,\ b > x_0 \\ a < x_0 \Rightarrow \dfrac{H(x_0)-H(a)}{x_0-a} = \underbrace{H'(c)}_{=0!} \quad \text{some } a < c < x_0 \\ \qquad\qquad \Rightarrow H(a) = C \quad \text{any } a \in I,\ a < x_0 \end{array}\right\} \begin{array}{l} H(x) = C \\ \forall x \in I ! \end{array}$$

So $H(x) = C$

so $G(x) - F(x) = C$ ■

Recall,

<u>Definition</u> $\int f(x)dx$ is Leibniz' notation (and ours too) for the set of all antiderivatives of $f(x)$, with respect to the variable x. We read this expression as "the integral of $f(x)$ $\underbrace{dx}$"
or "with respect to x"

<u>Exercise 4 a)</u> $\int x^3 dx =$

4b) $\int t^2 + 7t + 6 \; dt =$

Exercise 5

a) $\int 3x^7 - \frac{4}{x^2} + \frac{5}{\sqrt{x}}\,dx$

b) $\int 4\cos t + 8\sec^2 t + 3\csc t \cot t\,dt$

c) $\int u^n\,du$

$\int \cos u\,du$

$\int \sin u\,du$

d) $\int f'(x)dx$

e) $D_x\left(\int f(x)dx\right)$

Remark: (d), (e) show that differentiation and antidifferentiation are almost inverse procedures: (e) says that differentiation undoes antidifferentiation. (f) says that (up to an additive constant), antidifferentiation undoes differentiation.

Each differentiation rule can be read in the other direction to get an antidifferentiation rule:

Theorem B

(i) $\int k f(x)dx = k\int f(x)dx$ — const. mult. rule

(ii) $\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)dx$ — sum rule.

(iii) $\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$ — chain rule

(iv) $\int f\,g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)dx$ — product rule ("integration by parts" – 1220)

(v) $\int (g(x))^r g'(x)\,dx = \frac{1}{r+1}\,g(x)^{r+1} + C$ — generalized power rule, special case of (iii)

Exercise 6a) $\int (x^4+3x)^{30}(4x^3+3)\,dx =$

6b) Differentials, i.e. the method of substitution, often help us see how to use the chain rule backwards to find antiderivatives.
Try this in the problem above, using

$$u = x^4+3x$$
$$du = (4x^3+3)\,dx$$

6c) Try differential substitutions to solve

$$\int \frac{x}{\sqrt{x^2+4}}\,dx$$

Check your answer!

6d) $\int \sin^2(2x)\cos 2x \; dx$