Math 1210-3
Friday March 14

§ 3.7 Solving equations numerically

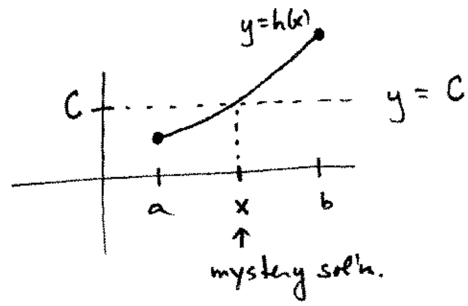
means using algorithms to get approximate numerical sol'ns, especially in cases where algebraic sol'ns cannot be found.

Setup: Suppose $h(x)$ is a function and we want a solution to the equation

$$h(x) = C$$

Suppose we also know

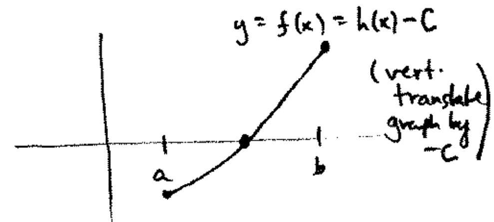
- $h(x)$ is continuous on the interval $[a,b]$
- $h(a) < C$ and $h(b) > C$
  (or $h(a) > C$ and $h(b) < C$




Step 1: This is the same as finding a root of the equation

$$f(x) = 0$$

where $f(x) := h(x) - C$




Bisection algorithm: Let $f(x)$ be continuous on $[a,b]$, with $f(a)$, $f(b)$ having opposite signs. The bisection algorithm finds approximate sol'ns to

$$f(x) = 0$$

as follows:

$x = a$ ; left guess
$X = b$ ; right guess
(1) $z = \frac{x+X}{2}$ ; midpoint of current left & right guesses
if $f(x)f(z) = 0$ return $z$ ; $f(z)$ will equal 0!
if $f(x)f(z) > 0$
then set $x = z$ and go to (1) ; move your left guess to the right, to $z$
if $f(x)f(z) < 0$
then set $X = z$ and go to (1) ; move your right guess, to the left, to $z$.

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Repeat a given number of times → after $n$ times you are within $\frac{b-a}{2^n}$ of a root, because each successive interval $[x, X]$ is a bisection of the previous one!

Advantages: bisection method always works to find (i.e. approximate) a sol'n to $f(x) = 0$ whenever $f$ is continuous.

Disadvantage: it's really slow.

Example : Approximate $\sqrt{2}$.

As you know (?), $\sqrt{2}$ is irrational!

If it was rational, then we could write

$\sqrt{2} = p/q$ $\quad$ p, q positive integers with no common pos. integer factors (besides 1)

$\Rightarrow 2 = p^2/q^2$

$\Rightarrow 2q^2 = p^2$

Thus $p^2$ is even, so p is also even.

Thus 2 divides p, so 4 divides $p^2$, $p^2 = 4m$

$\Rightarrow 2q^2 = 4m$

$\Rightarrow q^2 = 2m$

Thus $q^2$ is even so q is also even.

Thus 2 divides both p and q.

But we wrote $\sqrt{2} = p/q$ where p, q had no common prime factors.

OOPS! This is an inconsistency, or contradiction.

Thus, it is not possible to write $\sqrt{2} = p/q$, p, q integer.

Exercise 1 : To solve $x^2 = 2$ for $x > 0$

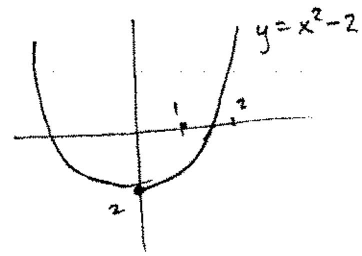
begin with

$[a, b] = [1, 2]$ $\quad$ since $1^2 = 1 < 2 < 4 = 2^2$

$f(x) = x^2 - 2$

Make a table of successive x, X values (and midpoints z, vals f(z))

| x | X | f(x) | f(X) | z | f(z) |
|---|---|---|---|---|---|
| 1 | 2 | −1 | +2 | 1.5 | .25 |
| 1 | 1.5 | | | | |
| | | | | | |
| | | | | | |




the computer takes over:

x (left) $\quad$ X (right)

1.375000000, 1.437500000
1.406250000, 1.437500000
1.406250000, 1.421875000
1.414062500, 1.421875000
1.414062500, 1.417968750
1.414062500, 1.416015625
1.414062500, 1.415039062 ← $1.414 < \sqrt{2} < 1.4151$

<u>Newton's method</u> uses Calculus to find roots of equations

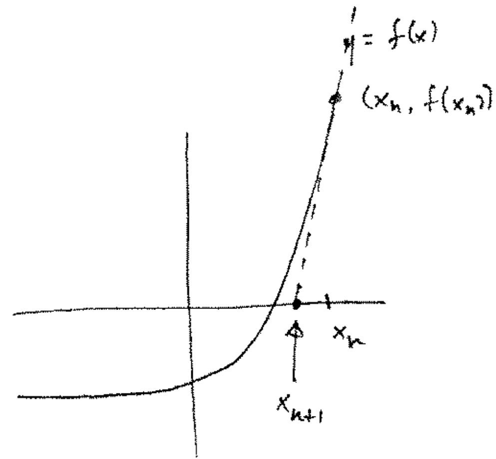
to find a solution $x$ to

$$f(x)=0$$

First find a "good" first guess $x_0$

Then get $x_{n+1}$ from $x_n$ as follows:

$x_{n+1}$ is the x-value at which the tangent line to $y=f(x)$, passing thru $(x_n, f(x_n))$, intercepts the x-axis.




details: tangent line has slope $f'(x_n)$

so eqtn is

$$y-f(x_n)=f'(x_n)(x-x_n)$$

If we set $y=0$ we get the following equation for the x-intercept:

$$-f(x_n)=f'(x_n)(x-x_n)$$

$$\boxed{x_n-\frac{f(x_n)}{f'(x_n)} = x := x_{n+1}} \longleftarrow \text{so } \boxed{x_{n+1}=g(x_n), \text{ with } g(x)=x-\frac{f(x)}{f'(x)}}$$

<u>Advantages</u>: <u>very</u> quickly approximates the root accurately.

<u>Disadvantages</u>: need a good first guess, or the method may not work. Also, requires that $f(x)$ be differentiable.

<u>Exercise 2</u>: Newton to approximate $\sqrt{2}$:

For $f(x)=x^2-2$

2a) Show the iteration function $g(x)$ for this example is

$$g(x)=\frac{1}{2}x+\frac{1}{x}$$

Fill in this table

| | |
|---|---|
| $x_0$ | 1 |
| $x_1$ | |
| $x_2$ | |
| $x_3$ | |

2b) Set $x_0=1$ and compute $x_1, x_2, x_3$ in table. (Use your calculator.) How close is $x_3$ to what your calculator says $\sqrt{2}$ equals?

Newton's method for $\sqrt{2}$.

```
> Digits:=40:  #we're going to get very close,
               #this is how many digits we'll use
> f:=x->x^2-2;  #a root of f is the square root of 2
  g:=x->x/2+1/x; #the iteration function in this
                 #example
```

$$f := x \to x^2 - 2$$

$$g := x \to \frac{1}{2}x + \frac{1}{x}$$

```
> x:=1.0; #initial guess
```

$$x := 1.0$$

```
> for i from 1 to 6 do
  x:=g(x); #iteration step
  print(x);
  od:
```

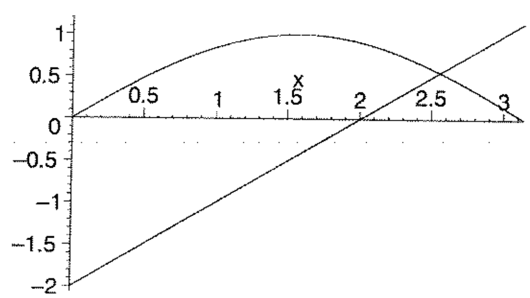
1.500000000000000000000000000000000000000
1.416666666666666666666666666666666666667
1.414215686274509803921568627450980392157
1.414213562374689910626295578890134910117
1.414213562373095048801689623502530243615
1.414213562373095048801688724209698078570

```
> sqrt(2.0);  #how close are we?
```

1.414213562373095048801688724209698078570

Example 3 page 193: find where the graphs of y=sin(x) and y=x-2 intersect:

```
> restart:
  with(plots):
Warning, the name changecoords has been redefined
> plot({sin(x),x-2},x=0..Pi,color=black);
      #we want the x-value of where curves cross
```




```
> f:=x->sin(x)-(x-2):  #we want a root of f
  g:=x->x-f(x)/(cos(x)-1.):  #iteration function
> x:=2.5:  #initial guess, from graph shown above
  for i from 1 to 3 do
      x:=g(x):
      print(x,sin(x)-(x-2));
  od:
```

2.554672011, -0.000872385
2.554195987, $-0.63\ 10^{-7}$
2.554195953, 0.

3 steps!