

Math 1210-3
Wed March 12

Finish graphing $y = \frac{-2x^2 - 5x + 1}{x+3}$ from yesterday.

Then...

§3.6 Mean Value Theorem for derivatives

This section helps explain earlier claims about inc/dec, cu/cd, and is also important for up-coming chapter.

Theorem (MVT). Let f be continuous on the closed interval $[a, b]$.
(Let f be differentiable on the open interval (a, b) .)

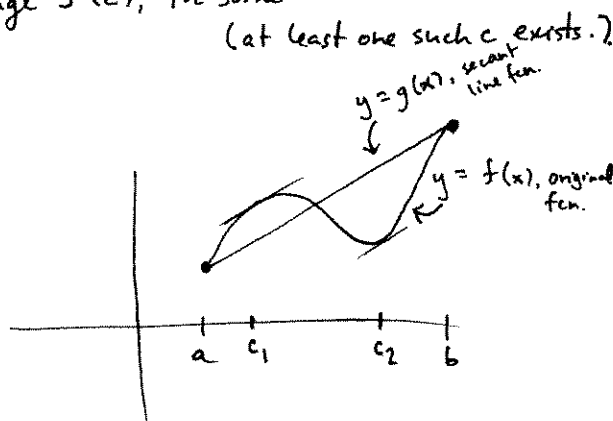
Then the average rate of change for $a \leq x \leq b$,

$$\frac{f(b) - f(a)}{b - a} \quad (\text{which we discussed in §3.1})$$

is actually the instantaneous rate of change $f'(c)$, for some $a < c < b$.
(at least one such c exists.)

Precisely: There exists (at least) one c , $a < c < b$
so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



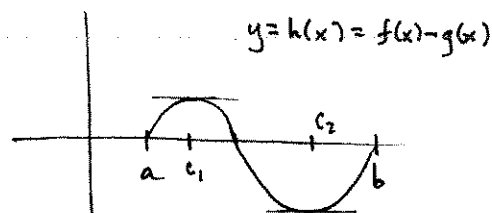
proof. Compare $f(x)$ to the
secant line through $(a, f(a))$ & $(b, f(b))$,
i.e. $y - f(a) = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$

So the secant line is the graph $y = g(x)$, for

$$g(x) = f(a) + \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

Consider $h(x) = f(x) - g(x)$

- h is continuous for $a \leq x \leq b$
- h is diff'ble for $a < x < b$, $h'(x) = f'(x) - \left(\frac{f(b) - f(a)}{b - a}\right)$.
- $h(a) = h(b) = 0$



If $h(x)$ has a positive max value at $x = c_1$, then $h'(c_1) = 0$
so $f'(c_1) = \frac{f(b) - f(a)}{b - a}$.

If $h(x)$ has a negative min value at $x = c_2$,
then $h'(c_2) = 0$, so
 $f'(c_2) = \frac{f(b) - f(a)}{b - a}$.

Otherwise $h(x) \equiv 0 \forall x \in [a, b]$,
so any $a < c < b$ satisfies
the claim. ■

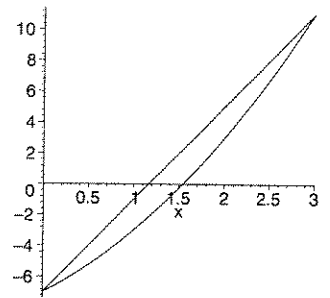
Illustrations of MVT

Exercise 1 : Suppose the highway patrol has timed you on a 40 mile stretch of IIS, and it took you $\frac{1}{2}$ hour to travel this distance. Can you claim you were never speeding?

Exercise 2 $f(x) = x^2 + 3x - 7$

$[a, b] = [0, 3]$

Find the average rate of change of f , and verify MVT

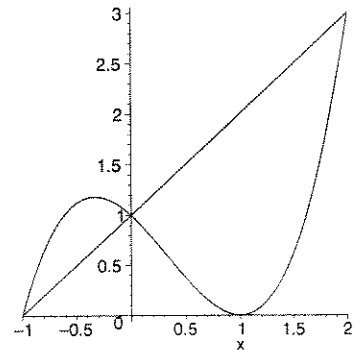


Exercise 3 $f(x) = x^3 - x^2 - x + 1$

$[a, b] = [-1, 2]$

Find average rate of change of f and verify MVT.

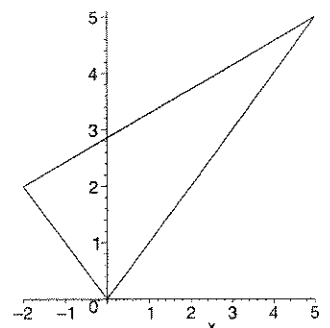
hint: $\frac{1+\sqrt{5}}{3} \approx 1.22$, $\frac{1-\sqrt{5}}{3} \approx -0.55$



Exercise 4 $f(x) = |x|$

$[a, b] = [-2, 5]$

What is c ?



Applications of MVT

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① To this chapter:

If f is continuous on $[a, b]$
& differentiable on (a, b)

and if $f'(x) > 0 \quad \forall x \in (a, b)$

then f is increasing on $[a, b]$. (This was half of our inc/dec theorem).

proof: Let $a \leq a_1 < b_1 \leq b$

We want to show $f(b_1) > f(a_1)$.

$$\text{But } \frac{f(b_1) - f(a_1)}{b_1 - a_1} = f'(c_1) > 0. \quad \text{Since } b_1 - a_1 > 0, \text{ this means } f(b_1) - f(a_1) > 0 \\ \text{i.e. } f(b_1) > f(a_1) \quad \blacksquare$$

(If $f'(x) < 0 \quad \forall x \in (a, b)$, the analogous argument shows f is decreasing)

② To the next chapter:

②a If f is continuous on $[a, b]$
& differentiable on (a, b)

and if $f'(x) = 0 \quad \forall x \in (a, b)$

Then $f(x)$ is a constant, $f(x) = C \quad \forall x \in (a, b)$.

proof. We'll show $f(x) = f(a) \quad \forall x \in (a, b)$:

$$\frac{f(x) - f(a)}{x - a} = f'(c) \quad a < c < x \\ = 0, \quad \text{so } f(x) = f(a) \quad \blacksquare$$

②b Let $F(x)$ and $G(x)$ be antiderivatives of the same function $h(x)$ on $[a, b]$,
then $G(x) = F(x) + C \quad \forall x \in [a, b]$, where C is a constant

proof: $F'(x) = h(x)$
 $G'(x) = h(x)$

$$\text{so } (G(x) - F(x))' = 0 \quad \forall x \in [a, b].$$

$$\text{so } G(x) - F(x) = C \quad \forall x \in [a, b], \text{ i.e. } G(x) = F(x) + C \quad \blacksquare \\ \text{by } \textcircled{2a}$$