Math 1210-3
Tuesday Jan 29

§1.2 Estimation, and the precise definition of limit.

The careful study of limits is very much related to the reasoning you use in error analysis for scientific experiments.

Consider an experiment to measure g, the acceleration of gravity: As we've worked out before, the distance a dropped object falls in time t is

$$(*) \qquad d = \frac{1}{2} g t^2$$

(neglecting friction, and until the object hits the ground.)

Experiment : Object dropped from height $10 \pm .002$ m
Time until it hits the ground is $1.43 \pm .01$ sec.

Exercise 1 : Use this data to estimate g, with error bounds.

Start with (*), and solve for

$$g = \frac{2d}{t^2} = \frac{2[10 \pm .002]}{(1.43 \pm .01)^2}$$

ans : $9.643 < g < 9.921$ m/s^2

(actual ans, $g \cong 9.81$ m/s^2)

In experiments, errors in the data "x" cause related errors in functions $f(x)$ that we compute from the data.
(On page 1, $g = g(d,t)$ depends on two pieces of data.)

By improving the accuracy of experimental data, "x" you improve the accuracy of $f(x)$.

The precise definition and usage of limits is essentially how a careful experimenter decides the necessary data precision in x which is required in order yield specified precision in $f(x)$:

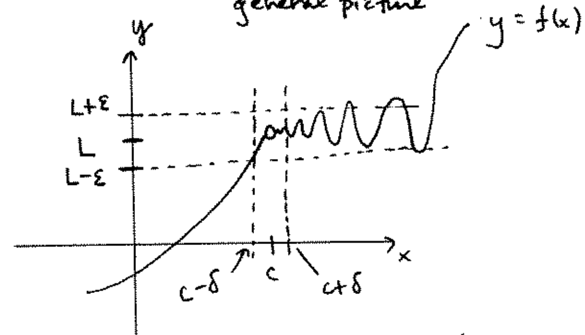
$\lim_{x \to c} f(x) = L$ means precisely that

for every $\varepsilon > 0$ there exists a $\delta > 0$ so that
"$\forall$" "$\exists$" "s.t."

$|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$

Limit definition

general picture



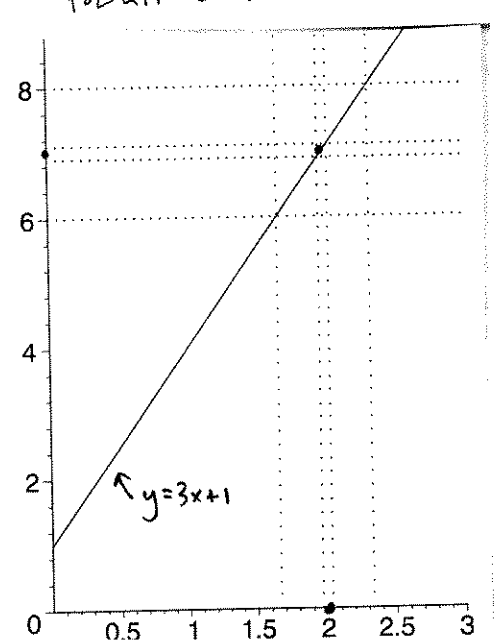
given $\varepsilon > 0$ (error tolerance) there should exist $\delta > 0$ so the graph of f lies within the horizontal strip $L - \varepsilon < y < L + \varepsilon$ for all $0 < |x - c| < \delta$.

Exercise 2: Use the ε-δ def. of limit (and page 2) to show

$$\lim_{x \to 2} 3x + 1 = 7$$

2a) Preliminary work

2b) Formal proof



The text works several examples very carefully, so if today's lecture leaves you feeling uneasy you may wish to study §1.2.

Exercise 3

3a) Find $\lim_{x \to 1} \dfrac{x^3 + x^2 + x - 3}{x - 1}$

3b) Prove your answer is correct using the ε-δ def.

Preliminary work	Formal proof

Exercise 4 On page 2 of Monday's (yesterday's) notes we used the Pythagorean Theorem to show

$$0 \leq (1 - \cos t)^2 + \sin^2 t \leq t^2$$

Use this estimate to give an ε-δ proof that

$$\lim_{t \to 0} \cos t = 1$$

or

$$\lim_{t \to 0} \sin t = 0$$

(your choice)

To save time in limit computations, it helps to know the following results. They are proven using the ε-δ definition, see §1.3 for some of the proofs

Theorem A (page 68) Main Limit Theorem

Let n be a positive integer, k a constant, f and g be functions that have limits at $x=c$ and let

Then

1. $\lim_{x\to c} k = k$
2. $\lim_{x\to c} x = c$
3. $\lim_{x\to c} k f(x) = k \lim_{x\to c} f(x)$
4. $\lim_{x\to c} (f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
5. $\lim_{x\to c} (f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
6. $\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$
7. $\lim_{x\to c} f(x)/g(x) = \lim_{x\to c} f(x) \Big/ \lim_{x\to c} g(x)$ provided $\lim_{x\to c} g(x) \neq 0$
8. $\lim_{x\to c} (f(x))^n = \left(\lim_{x\to c} f(x)\right)^n$
9. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ provided $\lim_{x\to c} f(x) > 0$ when n is even.

Exercise 5 Redo Exercise 3, using Theorem A.