Math 1210-3
Tuesday Jan. 15

P.4 antidifferentiation

• Begin with page 4 of Monday's notes

Then, **Theorem** (We'll prove this in §3.8 of Varberg, but it's believable.)

If $F(x)$ is an antiderivative of $f(x)$, then all other antiderivatives are of the form $F(x) + C$, where C is a constant.

i.e. $$\int f(x)\,dx = F(x) + C$$

Exercise 1

work backwards like in #6 of Monday notes, to compute the following integrals

1a) $\int x^3\,dx$

1b) $\int 4\,dx$

1c) $\int x^3 + 4\,dx$

1d) $\int 2x^5 + 3x^3 + x - 10\,dx$

1e) $\int -4t + 2\,dt$

1f) $\int t^2x^3\,dt$

1g) $\int t^2x^3\,dx$

Hopefully on page 1 we discovered

Theorem A If n is a non-negative integer, then

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{(where } C \text{ ranges over all constants)}$$

e.g.

$$\int 1\,dx =$$

$$\int x\,dx =$$

$$\int x^2\,dx =$$

$$\int x^3\,dx =$$

etc.

proof of Theorem A: Let $F(x) = \frac{x^{n+1}}{n+1}$. Then $F'(x) = \left(\frac{1}{n+1}\right)(n+1)\,x^n = x^n$.
This shows $F(x)$ is an antiderivative of x^n. You get every other antiderivative by adding (arbitrary) constants to $F(x)$. ∎

Theorem B If f and g are functions and "a" is a constant, then

(1) $\int a f(x)\,dx = a \int f(x)\,dx$

(2) $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

proof: If $F(x)$ is an antiderivative of $f(x)$, i.e. $F'(x) = f(x)$,
then $(aF)' = aF' = af$

so $aF(x)$ is an antiderivative of $af(x)$.

Thus
$$\int a f(x)\,dx = aF(x) + C_1 = a\,(F(x) + C) \qquad \left(C = \frac{C_1}{a}\right)$$
$$= a \int f(x)\,dx$$

Similarly, if $F' = f$ and $G' = g$, then
$$(F+G)' = f + g$$

So
$$\int f(x) + g(x)\,dx = F(x) + G(x) + C$$
$$= \int f(x)\,dx + \int g(x)\,dx$$ ∎

Exercise 2   Find the antiderivative of $x^2 - 1$ which has value 2 when $x = -2$

Exercise 3 : The acceleration due to gravity on the earth's surface is $g = -32$ ft/sec^2 $(= -9.8$ m/sec$^2)$

A ball is thrown straight upward with initial velocity of 128 ft/sec after which only gravity accelerates it. (We neglect friction.)

3a) What is the velocity $v(t)$, if $t = 0$ sec. is when object is released?

3b) When does ball object reach maximum height?

3c) If ball was thrown from initial height of 6 feet, how high does it go?

3d) When does it hit the ground?

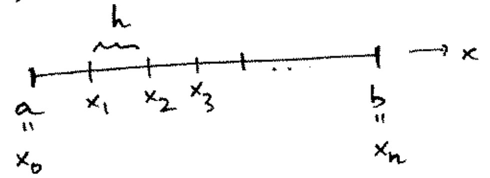
P.5 Fundamental Theorem of Calculus

The <u>definite</u> integral ("the integral from a to b of $f(x)\,dx$")

$$\int_a^b f(x)\,dx$$

is the following limit:

subdivide the interval $[a,b]$ into n subintervals, say of equal width $h = \frac{b-a}{n}$



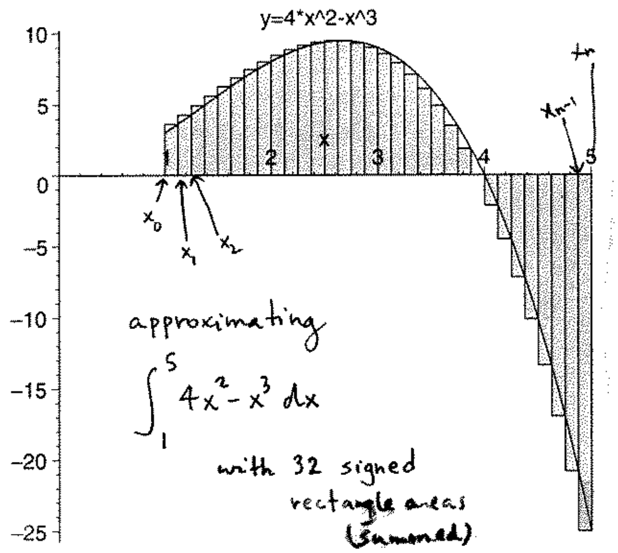
Compute

$$f(x_1)\underbrace{(x_1-x_0)}_{h} + f(x_2)\underbrace{(x_2-x_1)}_{h} + \dots + f(x_n)(x_n-x_{n-1})$$

$$= \sum_{i=1}^{n} f(x_i)\,h.$$

The limit of this Riemann sum as $h \to 0$ (i.e. as $n \to \infty$) exists if $f(x)$ is <u>continuous</u> on $[a,b]$, and is called

$\int_a^b f(x)\,dx$. <u>One</u> application is the compute the signed area between the graph of $f(x)$ and the x-axis, for $a \le x \le b$.



approximating $\int_1^5 4x^2 - x^3\,dx$ with 32 signed rectangle areas (summed)

The amazing FTC (I'll give a sketch of "why?!" tomorrow) says

<u>Fundamental Theorem of Calculus</u>

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antideriv. of f.

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> with(student):
  rightsum(4*x^2-x^3,x=1..5,1000);
  evalf(%);
  Int(4*x^2-x^3,x=1..5);
  evalf(%);
```

$n=1000$, $h = \frac{4}{1000} = .004 \rightarrow$ $\quad \frac{1}{250}\left(\sum_{i=1}^{1000}\left(4\left(1+\frac{i}{250}\right)^2 - \left(1+\frac{i}{250}\right)^3\right)\right)$

approx value $\rightarrow$ $\quad 9.277280000$

the limit $\rightarrow$ $\quad \int_1^5 4x^2 - x^3\,dx$

exact value $\rightarrow$ $\quad 9.333333333$

Exercise 4. Use FTC to compute $\int_1^5 4x^2 - x^3\,dx$

Compare to picture and computer output on preceding page

Exercise 5 Use geometry and the FTC to compute the area under $y = x+3$, between $x=0,2$

geometry:

FTC: