Name Solutions

Student I.D. ______________________

Math 1210-3
Final Exam
April 30, 2008

Please show all work for full credit. This exam is closed book and closed note, except for the single customized 4 by 6 inch index card you have been allowed to bring. (Two Varberg formula pages are at the end of the exam.) You may also use a scientific calculator, but not a graphing calculator or one which can compute derivatives and integrals. There are 150 points possible, as indicated below and in the exam. You have two hours to complete the exam, so apportion your time accordingly. Good Luck!!

	Score	POSSIBLE
1		20
2		15
3		20
4		15
5		15
6		10
7		15
8		30
9		10
TOTAL		150

1) Compute the following limits. Limits which equal infinity (or -infinity) are possible. It is also possible that the limit does not exist, in which case you should answer DNE, and indicate your reasoning.

1a) $\lim_{x\to 3} \frac{x^2-x-6}{x^3-27}$ $\quad \frac{0}{0}$ $\quad = \lim_{x\to 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x^2+3x+9)} = \frac{5}{9+9+9} = \boxed{\frac{5}{27}}$

(5 points)

$x^2-x-6 = (x-3)(x+2)$

$x^3-27 = (x-3)(x^2+3x+9)$

$$\begin{array}{r|l} & x^2+3x+9 \\ \hline x-3 & x^3-27 \\ & x^3-3x^2 \\ \hline & 3x^2-27 \\ & 3x^2-9x \\ \hline & 9x-27 \\ & 9x-27 \\ \hline & 0 \end{array}$$

1b) $\lim_{x\to 2+} \frac{x+8}{x^2-4}$

(5 points)

$\lim_{x\to 2^+} \frac{x+8}{(x-2)(x+2)} = \frac{10}{0^+(4)} = \boxed{+\infty}$

1c) $\lim_{x\to 0} \frac{3x}{|x|}$

(5 points)

$\left.\begin{array}{l} \lim_{x\to 0^+} \frac{3x}{x} = 3 \\ \lim_{x\to 0^-} \frac{3x}{(-x)} = -3 \end{array}\right\}$ So, since the 1-sided limits are not equal,

$\lim_{x\to 0} \frac{3x}{|x|}$ $\boxed{\text{DNE}}$

1d) $\lim_{h\to 0} \frac{(5+h)^3-125}{h}$

(This is a derivative in disguise, so a short cut is available!)

(5 points)

$= D_x x^3, @ \ x=5 \quad ; \quad = 3x^2, @ \ x=5$

$\left(\lim_{h\to 0} \frac{f(5+h)-f(5)}{h}\right)$ $\quad = 3\cdot 25$

$\boxed{=75}$

long way: $\lim_{h\to 0} \frac{\cancel{5^3} + 3\cdot 5^2\cdot h + 3\cdot 5\cdot h^2 + h^3 - \cancel{125}}{h}$

$= \lim_{h\to 0} 75\frac{h}{h} + 15\frac{h^2}{h} + \frac{h^3}{h} = \lim_{h\to 0} 75 + 15h + h^2 = 75$ ✓

2) Compute the following derivatives:
2a)

$-6t^{-1}$

$$D_t\left(\pi^2 + 3t^2 - \frac{6}{t}\right)$$

(5 points)

$= 0 + 6t - 6(-1)t^{-2}$

$\boxed{= 6t + \frac{6}{t^2}}$

2b)

$$D_x[\sec(x)^2 - \tan(x)^2]$$

(5 points)

$\sec^2 x - \tan^2 x = \frac{1}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} = \frac{1-\sin^2 x}{\cos^2 x} = \frac{-\cos^2 x}{\cos^2 x} = -1$

So $D_x(\sec^2 x - \tan^2 x) = D_x(-1) = \boxed{0}$

(or, $D_x(\sec^2 x - \tan^2 x) = 2\sec x(\sec x \tan x) - 2\tan x(\sec^2 x) = \boxed{0}$).

2c) f′ (x) for

$$f(x) = \frac{1}{\sqrt{x^3+4}}.$$

(5 points)

$f(x) = (x^3+4)^{-1/2}$

$f'(x) = -\frac{1}{2}(x^3+4)^{-3/2} \cdot 3x^2$

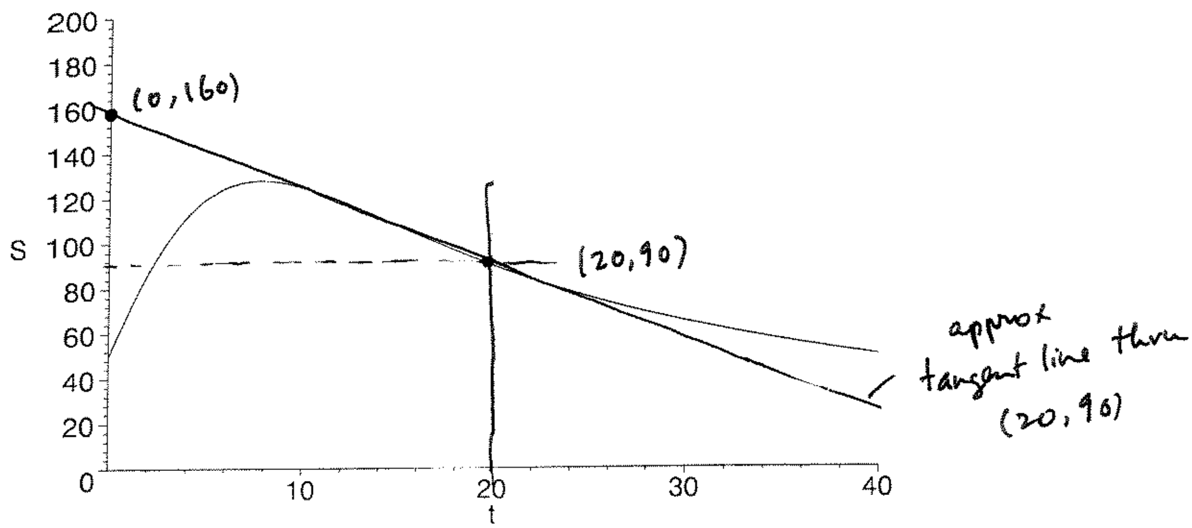
$\boxed{= -\frac{3}{2}x^2(x^3+4)^{-3/2}}$

or, quotient rule,

$f'(x) = \frac{0 - 1 \cdot \frac{1}{2}(x^3+4)^{-1/2} 3x^2}{x^3+4}$

$= -\frac{3}{2}(x^3+4)^{-3/2} \cdot x^2$

3) Mathville is hit with a spring flu epidemic! The number S(t) of sick people at time t days after the epidemic begins is plotted below:



3a) Using the graph above, estimate how many people are sick on day t=20. (It might be helpful to fold your paper and use the edges to draw any lines you need.) Show work!

(5 points)

about $\boxed{\text{90 people}}$ (80-100 o.k.)

3b) Using the graph above, estimate how fast the number of sick people is decreasing on day t=20. Show work!

(5 points)

$$S'(20) \approx \frac{\text{rise}}{\text{run}} = \frac{90-160}{20-0} = -\frac{7}{2} \text{ people/day}$$

(so the number is decreasing at about $\boxed{7/2 \text{ people/day}}$)

3c) It turns out the sick people graph shown above is very close to the graph of the function (3.5) (3-4 o.k.)

$$S(t) = \frac{20\,t + 50}{1 + 0.01\,t^2}.$$

Use this formula for S(t) to calculate values for how many people are sick on day t=20, and also to calculate how fast the number of sick people is decreasing on day 20. (You should get numbers close to the ones you found geometrically in parts (3a) and (3b) !)

(10 points)

$$S(20) = \frac{20 \cdot 20 + 50}{1 + .01(400)} = \frac{450}{5} = \boxed{90 \text{ people}}$$

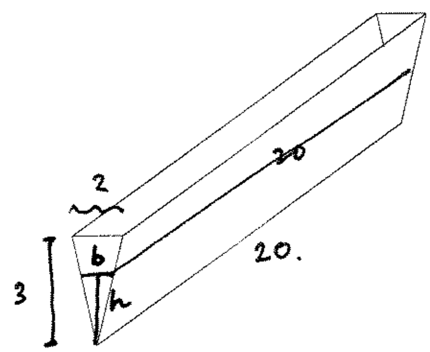
$$S'(20) = S'(t)\Big|_{t=20} = \frac{20(1+.01t^2) - (20t+50)(.02t)}{(1+.01t^2)^2}\Bigg|_{t=20}$$

$$= \frac{20(5) - 450(.4)}{(1+4)^2}$$

$$= \frac{100-180}{25} = -\frac{80}{25} = -\frac{16}{5} = \boxed{-3.2}$$

$\boxed{\text{so, dec. @ 3.2 people/day}}$

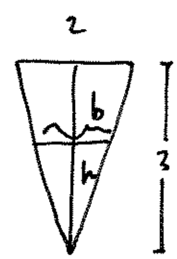
4) A drinking trough for cattle has a cross section in the form of an inverted isoceles triangle. The base of the triangle is 2 feet long and its height is 3 feet, as shown below. The trough is 20 feet long.



If water is filling the trough at a constant rate of 4 cubic feet per minute, then how fast is the water depth increasing when the depth is 2 feet? Let V = vol of water, h = depth, b = base, all fcns of t.
(15 points)

Question: If $V'(t) = 4$ find $h'(t)$ when $h = 2$

$$V = A \cdot 20$$
$$= \frac{1}{2} bh\, 20$$
$$V = 10bh$$
$$= 10\left(\frac{2}{3}\right)h^2$$
$$V = \frac{20}{3}h^2$$



$\frac{b}{h} = \frac{2}{3}$ so $b = \frac{2}{3}h$ at all t.

$\frac{D}{t}$:

$$V'(t) = \frac{40}{3} h h'(t)$$

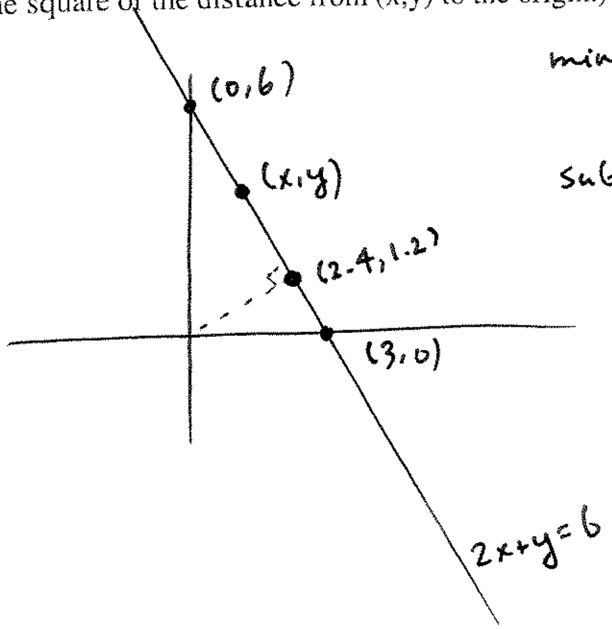
@ $h = 2$ we have $V'(t) = 4$, so

$$4 = \frac{40}{3} \cdot 2 \cdot h'(t)$$
$$1 = \frac{20}{3} h'$$

$$\boxed{h'(t) = \frac{3}{20} \text{ ft/min} = .015}$$

5) Use Calculus to find the point on the line $2x + y = 6$ which is closest to the origin. (Hint: minimize the square of the distance from (x,y) to the origin.)

(15 points)



minimize

$$d^2 = x^2 + y^2$$

subject to $2x + y = 6$

$$y = 6 - 2x$$

minimize

$$f(x) = x^2 + (6-2x)^2$$

$$f'(x) = 2x + 2(6-2x)'(-2)$$

$$= 2x + 8x - 24$$

$$= 10x - 24$$

$$= 10(x - 2.4)$$

$f'(x) = 0$ @ $x = 2.4$

(and $f''(x) = 10 > 0$ so f is a CU parabola, and $f(2.4)$ is a minimum)

$x = 2.4$

$y = 6 - 4.8 = 1.2$

$\boxed{\text{Nearest point is } (2.4, 1.2)}$

notice, the line from (0,0) to (2.4, 1.2) has slope ½, which is the negative reciprocal of −2, the slope of the original line $2x + y = 6$. This makes sense geometrically, since we expect the line to the nearest point to be perpendicular to the original line.

6) The following table shows the depth of a river at a certain cross-section perpendicular to the water flow. The river is 20 feet wide at this point. The depths y in feet, at position x feet from one shoreline are given by

x (location)	0	5	10	15	20
y (depth)	0	3	6	4	2

6a) What is the Simpson's (parabolic) rule approximation to the cross-sectional area of the river at the location above, with n=4 subdivisions of x-interval [0,20]?

(5 points)

$$\int_0^{20} f(x)dx \approx \frac{b-a}{3n}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$ parabolic rule.

$$= \frac{20}{12}\left(0 + 4\cdot 3 + 2\cdot 6 + 4\cdot 4 + 1\cdot 2\right)$$

$$= \frac{5}{3}\left(12 + 12 + 16 + 2\right)$$

$$= \frac{5}{3}(42) = 5\cdot 14 = \boxed{70\ ft^2}$$

6b) Use your answer to part (6a) to approximate the rate at which water is flowing down the river, assuming that the water velocity at this cross-section is approximately two feet per second. Make sure to include your units as part of your answer.

(5 points)

in 1 second, a cylinder of crossectional area 70 ft^2 & length 2 feet will have passed by the cross section,

so flow rate is

$$70\cdot 2 = \boxed{140\ ft^3/sec.}$$

7) Compute the following integrals:

7a) $\int_{-4}^{4} 4x - 12x^2 + 5\sin(3x)\,dx$ $= 2\int_0^4 -12x^2\,dx$ by symmetry

odd odd even

symmetric interval

(5 points)

$= 2\left(-4x^3\right)\Big]_0^4$

$= 2\cdot(-4^4) = \boxed{-512}$

7b) $\int_0^1 x(3x^2-1)^2\,dx$ or $\frac{1}{18}(3x^2-1)^3\Big]_0^1 =$ same as below.

(5 points)

$u = 3x^2 - 1$

$du = 6x\,dx$

$= \frac{1}{6}\int_0^1 (3x^2-1)^2\, 6x\,dx = \frac{1}{6}\int_{-1}^{2} u^2\,du = \frac{1}{18}u^3\Big]_{-1}^{2} = \frac{1}{18}(8+1) = \boxed{\frac{1}{2}}$

$x = 0 \quad x = 1$

$u = -1 \quad u = 2$

7c) $\int_0^{\frac{\pi}{8}} 10\cos(4t)\,dt$ or $\frac{10}{4}\sin 4t\Big]_0^{\pi/8}$

(5 points)

$= \frac{5}{2}(1-0) = \frac{5}{2}$

$u = 4t$

$du = 4\,dt$

$t=0,\ u=0$

$t=\pi/8,\ u=\pi/2$

$= \frac{10}{4}\int_0^{\pi/8} \cos 4t \;\; 4\,dt$

$= \frac{5}{2}\int_0^{\pi/2} \cos u\,du$

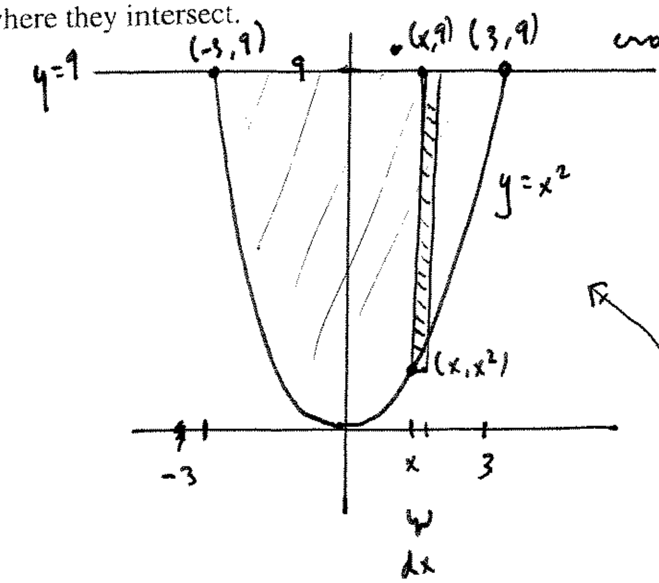
$= \frac{5}{2}\sin u\Big]_0^{\pi/2}$

$= \frac{5}{2}(1-0)$

$\boxed{= \frac{5}{2}}$

8a) Sketch the region between the graphs of $y=x^2$ and $y=9$. Make sure to label the curves and the points where they intersect.

(5 points)



cross when $9=x^2$
$x=\pm 3$

8b) Find the area of the region you sketched in 8a).

(5 points)

$dA=(9-x^2)\,dx$ (see rectangle in 8a picture)

$$A=\int_{-3}^{3} 9-x^2\,dx$$

$$=2\int_{0}^{3} 9-x^2\,dx$$

$$=2\left[9x-\frac{x^3}{3}\right]_0^3$$

$$=2\left[27-\frac{27}{3}\right]$$

$$=2[18]$$

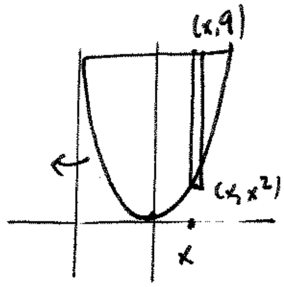
$$\boxed{=36}$$

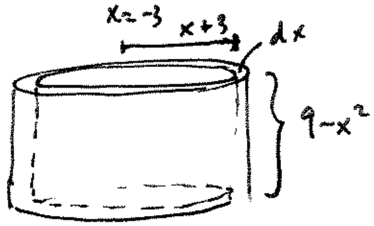
solid object resemble a bundt cake.

8c) Use planar slabs or cylindrical shells (your choice!) to find the volume of revolution obtained by rotating the region in (8a) about the vertical line $x = -3$.

(15 points)

shells easiest





$$dV = 2\pi(x+3)(9-x^2)\,dx$$
$$= 2\pi(-3x^2 - x^3 + 9x + 27)\,dx$$
$$V = 2\pi\int_{-3}^{3} -3x^2 - x^3 + 9x + 27\,dx$$

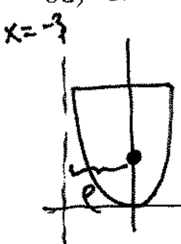
$$= 2\pi\cdot 2\int_0^3 -3x^2 + 27\,dx \quad \text{(by even/odd symmetry)}$$
$$= 4\pi\left[-x^3 + 27x\right]_0^3$$
$$= 4\pi\left[\overset{-27+27\cdot 3}{\cancel{+8+84}}\right] = 4\pi[54]$$

$\cancel{= 184\pi}$

$$\boxed{= 216\pi}$$

8d) Check your answer in 8c), by using Pappus' Theorem to recompute the volume of revolution.

(5 points)



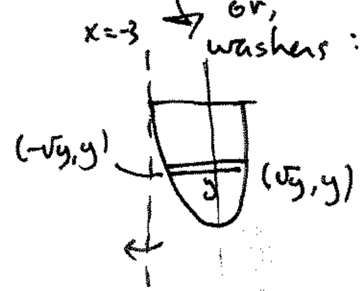
$\bar{x} = 0$ by symmetry.
$\bar{y} = ?$

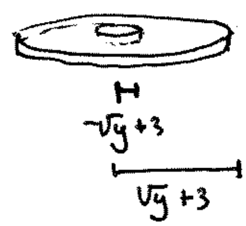
Pappus': $V = 2\pi\rho A$, ρ = dist. from center of mass to rotation axis

$$= 2\pi\cdot 3\cdot 36 = \boxed{216\pi}$$

$\uparrow\ \rho$ $\quad \uparrow\ A$ from 8a)

or, washers:





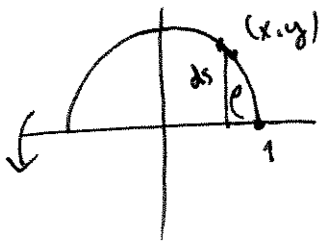
$$dV = \pi\left((\sqrt{y}+3)^2 - (-\sqrt{y}+3)^2\right)dy$$
$$= \pi\left(\cancel{y} + 6\sqrt{y} + \cancel{9} - (\cancel{y} - 6\sqrt{y} + \cancel{9})\right)dy$$
$$dV = 12\pi y^{1/2}\,dy$$
$$V = \int_0^9 12\pi y^{1/2}\,dy = 12\pi\,\frac{2}{3}y^{3/2}\Big]_0^9$$
$$= 8\pi\cdot 27 = \boxed{216\pi}$$

9) The parametric curve

$$x = \cos(t)$$
$$y = \sin(t)$$

describes the upper half of the unit circle, for $0 \le t \le \pi$. When this curve is rotated about the x-axis it generates a sphere of radius 1. Use a definite integral for surfaces areas of revolution, to verify that this sphere has area 4π.

(10 points)



$dA = 2\pi\rho \, ds$

$= 2\pi y \, ds$

$= 2\pi \sin t \sqrt{(dx)^2+(dy)^2}$

$= 2\pi \sin t \underbrace{\sqrt{\sin^2 t + \cos^2 t}}_{1} \, dt$

$dx = -\sin t \, dt$
$dy = \cos t \, dt$

So $dA = 2\pi \sin t \, dt$

$A = \int_0^{\pi} 2\pi \sin t \, dt$

$= 2\pi \left(-\cos t\right)\Big]_0^{\pi}$

$= 2\pi \left(--1 - -1\right)$

$\boxed{A = 4\pi}$ ✓