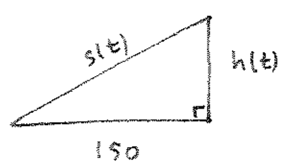
Math 1210-3
Tuesday Feb. 19

§2.8 Related rates + geometry review

the set up is that you are dealing with several related functions of time, and you wish to deduce how fast one of them is changing at a particular instant, from information about the function values and how fast the others are changing – this is an important use of the chain rule

For example,

Exercise 1 (p. 135) A balloon is released 150′ away from an observer and rises 8′/sec (′ means feet)
When the height of the balloon is 50′, how fast is the distance from the observer changing?

① 

④

②

⑤

③

steps
① construct diagram with relevant functions (and constants) labeled
② what is known, what is wanted? express mathematically
③ relate the known & and unknown functions with an equation (or more than one equation)
④ differentiate with respect to time, in ③
⑤ substitute known function values and rates into ④; deduce desired result.

practice!

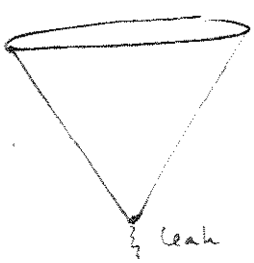
Exercise 2 a disk is growing, and when $r = 10$ cm the radius is increasing at a rate of 2 cm/min

How fast is the area growing at that instant?

Exercise 3 A conical tank is leaking water. The tank is 12 ft high, its horizontal cross sections are disks, and the disk at the top has radius 6 ft.

Suppose that when the water depth is 8 feet it is decreasing at a rate of $1/10$ ft/min.

How fast is water leaking out of the tank?



GEOMETRY REVIEW

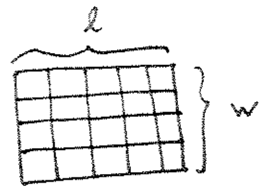
We'll do more related rates problems tomorrow,
but now is a good time to review the geometry of basic measurements.
(This will also help for optimization material in chapter 3.)

(A) Basic length, area, volume (or, why addition, multiplication was discovered/invented)

addition

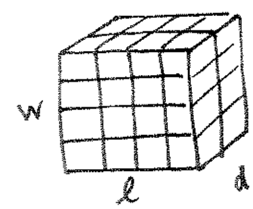
length
$L = l$ units
e.g. cm^1
ft, etc.

multiplication



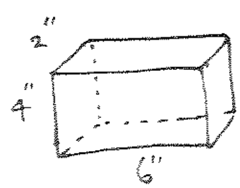
area
$A = lw$ square units
e.g. cm^2
ft^2

more multiplication



volume
$V = lwd$ cubic units
e.g. cm^3
ft^3

Exercise 4: A rectangular block has dimensions 2 in, 4 in, 6 in.



4a) Find the volume of the block

4b) Find the total surface area of all 6 faces

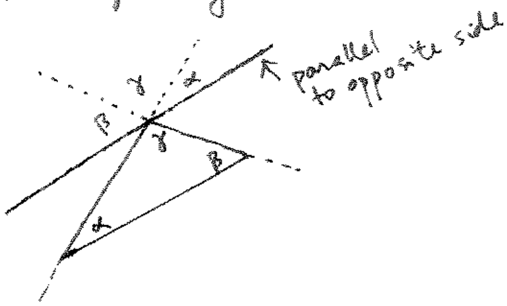
4c) Find the total edge length, of all 12 edges.

Very important: Notice that every formula for length, area, volume that we derive in the following pages has the correct units!!, either linear, square, or cubic. Knowing this will help you remember the correct formulas

linear → length, square → area, cubic → volume.

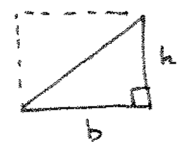
(B) Triangles

First, recall why the angles in every triangle always add up to π radians ($\frac{1}{2}$ revolution)
– it's just the geometry of transversals!



Second, $A = \frac{1}{2}bh$ for a right triangle, since it's half a rectangle:

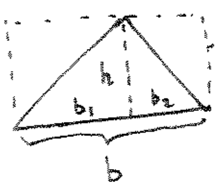
base choice — corresponding height



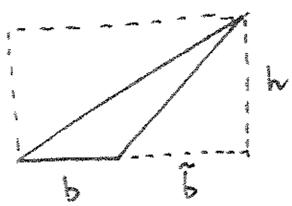
$A = \frac{1}{2}bh$

Exercise 5 Why is the area of every triangle $A = \frac{1}{2}bh$?

two cases:

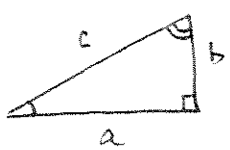


or



Exercise 6 Explain why the Pythagorean Theorem is true!! This is really an area computation which yields a statement about lengths, which is why so many people think it's "magic".

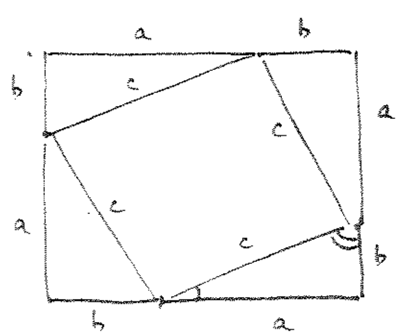
Hint: Use facts from this page to compute the total area of the square below, using two correct computations which must therefore give equal results.



$a^2 + b^2 = c^2$

↑

this is the statement of the Pythagorean Thm (which we often write $c = \sqrt{a^2+b^2}$)

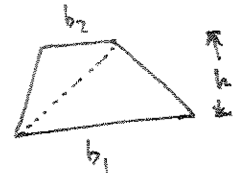


↑

this square contains a proof!!

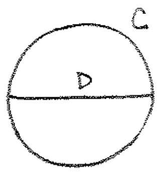
Exercise 7 : You can find the area of any polygon by decomposing it into triangles.
Explain why

7a) Area of trapezoid is $A = \left(\frac{b_1 + b_2}{2}\right) h$



7b) Area of parallelogram is $A = bh$

(C) Disks and circles.

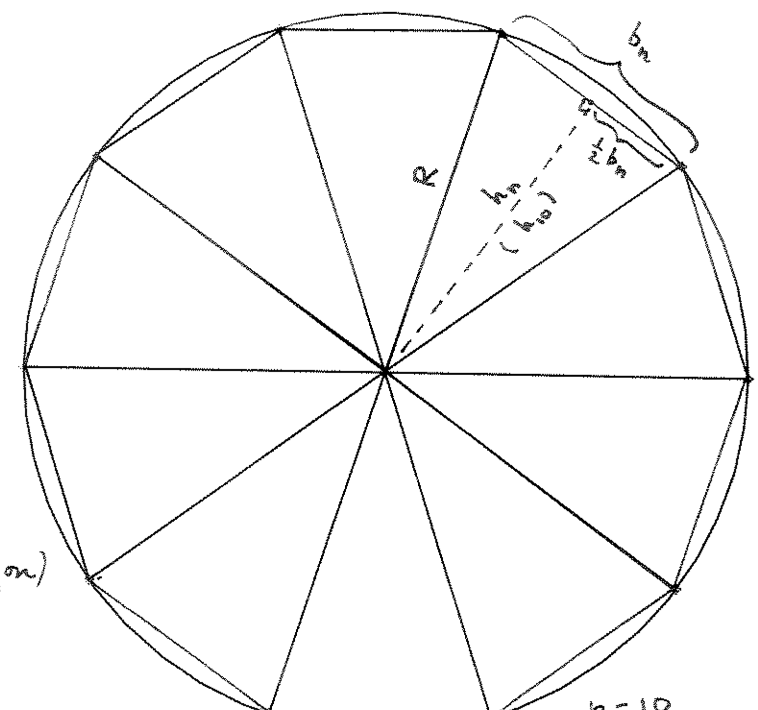


π is defined as $\frac{C}{D}$, C = circumfrence, D = diameter. This ratio is independent of the size of the circle

so since $D = 2R$,

$\boxed{C = 2\pi R}$

Exercise 8 Inscribe "n" congruent triangles inside a radius R disk, as shown for n=10 at the right. Let A_n be the area of the inscribed n-gon, i.e. n times the area of each triangle. Let P_n be the perimeter of the n-gon, i.e. n times the base of each triangle



n=10
triangles inscribed in a circle

8a) $\lim_{n\to\infty} P_n = \lim_{n\to\infty} n b_n = C$
$= 2\pi R$

8b) Area of disk $A = \lim_{n\to\infty}$ (area of n-gon)

$= \lim_{n\to\infty}$

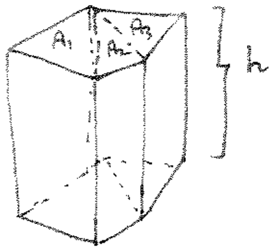
Explain why this leads to

$\boxed{A = \pi R^2}$

(Note too, $\frac{dA}{dR} = C$?!)

(D) 3-dimensional objects

cylinders:



(cross section area $\mathcal{B}$, here $\mathcal{B} = A_1 + A_2 + A_3$)

Any right-vertical cylinder with all horizontal cross-sections identical, has volume

$$\boxed{V = \mathcal{B}h}$$

($\mathcal{B}$ = cross-section area)

(This follows by adding "elementary" volumes of triangular prisms which partition the cylinder)

& If the perimeter of the cross section is P, then the area of the sides is

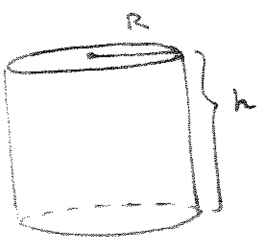
$$\boxed{A = Ph}$$

This is because if you wrap a vertical piece of paper around the object, a rectangle of length P & height h covers the sides exactly.

Exercise 9 Consider a solid cylinder with circular cross sections, radius R; and height h.

9a) What is volume V?

9b) What is the total surface area? (of its outside)

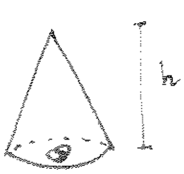


Balls, spheres, cones: These formulas can be derived by "elementary" arguments, at least the ancient Greeks knew how. We'll check them later in the course using Calculus. For now, notice that the units help you remember the formulas

solid radius R ball has volume $V = \frac{4}{3}\pi R^3$ cubic units!

the sphere shell has area $A = 4\pi R^2$ square units!

$\left(\frac{dV}{dR} = A \text{ too}\right)$

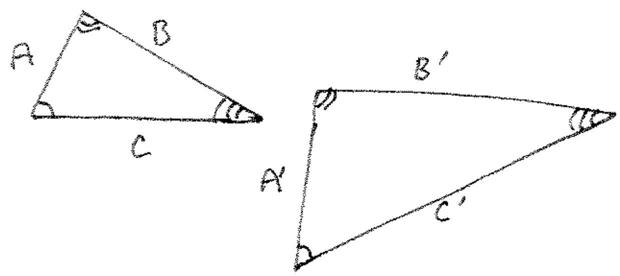


Solid cone with base area $\mathcal{B}$ and height h has volume

$V = \frac{1}{3}\mathcal{B}h$ cubic units

E Rescaling

triangles: two triangles are <u>similar</u> if they have the same interior angles. This means the triangles are rescalings of eachother, possibly combined with a reflection

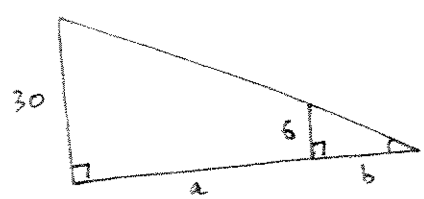


for similar Δ's, corresponding ratios of side-lengths agree:

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'},$$

so also $\frac{A}{B} = \frac{A'}{B'}$ etc.

<u>Exercise 10</u> a) How are a, b related?



10b) How are r and h related?