Math 1210-3
Friday Feb. 15

Left-overs from Wednesday:

3b) Find $\frac{dy}{dx}$ for $y = \left(\frac{\sin x}{\cos 2x}\right)^3$

4) If $y = \sin 2t$, find

4a) y'

4b) $D_t^2 y$

4c) $\frac{d^3y}{dt^3}$

4d) $y^{(10)}(t)$

§ 2.7 Implicit differentiation

warmup:

Exercise 1 : We've already checked the power rule $D_x x^n = n x^{n-1}$

for $n = 0, 1, 2, ..$ (polynomial calc.)
$n = -1, -2, -3$ (quotient rule.)

In fact, as was mentioned on Wednesday, the power rule holds for any rational number power. Let's check for $n = \frac{1}{2}$.

1a) Check $D_x x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}}$, using the limit definition (and "Δx" instead of "h")

$$D_x x^{\frac{1}{2}} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$$

1b) More clever way:

If $y = f(x) = \sqrt{x}$

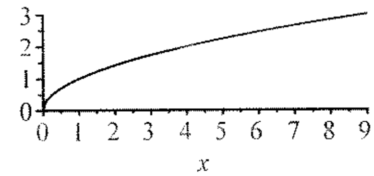
then $(f(x))^2 = x$

so $D_x (f(x))^2 = D_x x$

$2 f(x) f'(x) = 1$

$f'(x) = \frac{1}{2 f(x)} = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2}$!

$y = \sqrt{x}$



(upper half of the parabola $y^2 = x$)

This more clever way is an example of <u>implicit differentiation</u>:

If $y = f(x)$ (or, we often write $y = y(x)$)
satisfies an equation (like $y^2 = x$ above),
we can differentiate both sides of the equation to find y'.

<u>Example</u> Let $y = x^n = x^{p/q}$ where p, q are integers (and $x > 0$ if q is even)

$= (\sqrt[q]{x})^p$

(the algebra to make a direct proof like 1a) would be <u>really</u> hard!)

Then
$(y(x))^q = x^p$

D_x:

$q y^{q-1} y' = p x^{p-1}$ (chain rule!)

$y' = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}}$

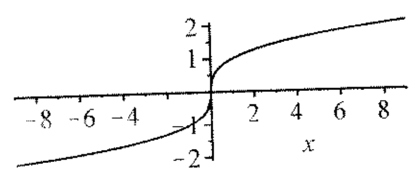
$y' = \frac{p}{q} \frac{x^p}{y^q} \frac{x^{-1}}{y^{-1}}$

$= \frac{p}{q} \cdot 1 \cdot \frac{y}{x}$

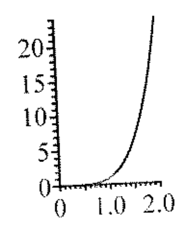
$y' = \frac{p}{q} \frac{x^{p/q}}{x} = \frac{p}{q} x^{p/q - 1}$

$y' = n x^{n-1}$

power rule works for rational powers too!



$y = \sqrt[3]{x}$, sideways cubic $y^3 = x$



$y = x^{55/12}$

<u>Exercise 2</u>: $D_x \left(\sqrt[3]{\cos x} \right) =$

You can use implicit differentiation (and it's especially useful) when it's not even possible to find an explicit formula for $y = f(x)$.

Exercise 3 Consider the graph of the equation $x^3 + y^2 + x^4y^4 = 3$

3a) Verify that $(1,1)$ is on the graph, algebraically.

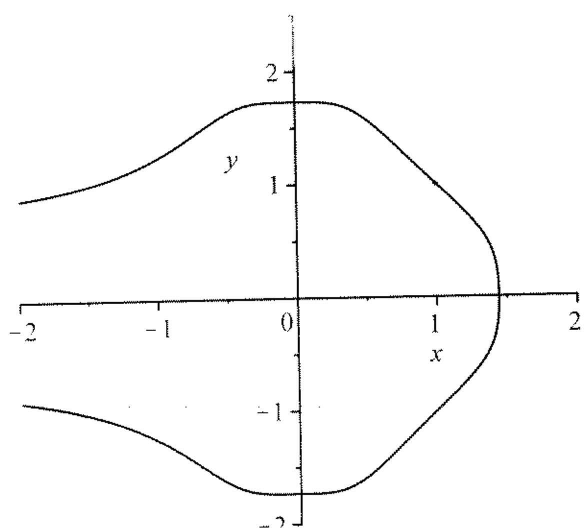
3b) From the plot it appears that the portion of the graph above the x-axis is the graph of some function $y = f(x)$, even though the explicit formula would be messy.

Use implicit differentiation of the equation

$$x^3 + y^2 + x^4y^4 = 3$$

to find y' $(= f'(1))$ at $(x,y) = (1,1)$.
Compare to your "folding paper" estimate

$x^3 + y^2 + x^4y^4 = 3$



3c) What is the slope of the graph $x^3 + y^2 + x^4y^4 = 3$ at $(1,-1)$?