Name SOLUTIONS

Student I.D. ____________________

Math 1210-3
Exam #3
April 4, 2008

Please show all work for full credit. This exam is closed book, closed note, but you may use a scientific (non-graphing) calculator. There are 100 points possible, as indicated below and in the exam. Since you only have 60 minutes you should be careful to not spend too long on any one problem. Good Luck!!

Score	POSSIBLE
1	25
2	20
3	30
4	25
TOTAL	100

1) Compute the following:

1a) $\int \left(2\,t^3 - \frac{3}{\sqrt{t}} + 16\cdot cost\right)\,dt$

(6 points)

$$\int 2t^3 - 3t^{-1/2} + 16\cos t \;\;dt$$

$$= \frac{2}{4}t^4 - 3\,\frac{t^{1/2}}{1/2} + 16\sin t + C$$

$$\boxed{= \frac{1}{2}t^4 - 6t^{1/2} + 16\sin t + C}$$

1b) $\int_0^2 \frac{3\,x}{\sqrt{2\,x^2+1}}\,dx$

$$\frac{3}{4}\int_0^2 (2x^2+1)^{-1/2}\;4x\,dx$$

substitute

(8 points)

$u = 2x^2 + 1$

$du = 4x\,dx$

$x = 0 \rightarrow u = 1$

$x = 2 \rightarrow u = 9$

$$\frac{3}{4}\int_1^9 u^{-1/2}\,du$$

$$= \frac{3}{4}\left(2u^{1/2}\right)\Big]_1^9 = \frac{3}{4}(6-2) = \boxed{3}$$

1c) $D_x\left(\int_0^{4x} \underbrace{t^2\sin(t^3)}_{f(t)}\,dt\right) = D_x\Big(F(4x) - F(0)\Big)$ where $F'(t) = f(t)$

(6 points)

$$= F'(4x)\cdot 4 - 0$$

$$= f(4x)\cdot 4$$

$$= 16x^2(\sin(64x^3))\cdot 4$$

$$\boxed{= 64x^2\sin(64x^3)}$$

you can also do this the long way, i.e. by first finding the definite integral explicitly, then computing D_x of it

long way:

$$\frac{1}{3}\int_0^{4x} 3t^2\sin t^3\,dt$$

$u = t^3$

$du = 3t^2\,dt$

$t = 0,\ u = 0$

$t = 4x,\ u = 64x^3$

$$\frac{1}{3}\int_0^{64x^3}\sin u\,du$$

$$\frac{1}{3}(-\cos u)\Big]_0^{64x^3} = -\frac{1}{3}\cos(64x^3)$$

$$D_x(\%)$$

$$= -\frac{1}{3}(-\sin 64x^3)\cdot 64\cdot 3x^2$$

$$= 64x^2\sin(64x^3)$$

✓

1d) $\int_1^2 4\,f(x)\,dx$, given that $\int_{-1}^2 f(x)\,dx = 3$ and $\int_{-1}^1 f(x)\,dx = 6$.

(5 points)

$$\int_{-1}^2 f(x)\,dx = \int_{-1}^1 f(x)\,dx + \int_1^2 f(x)\,dx$$ interval additivity

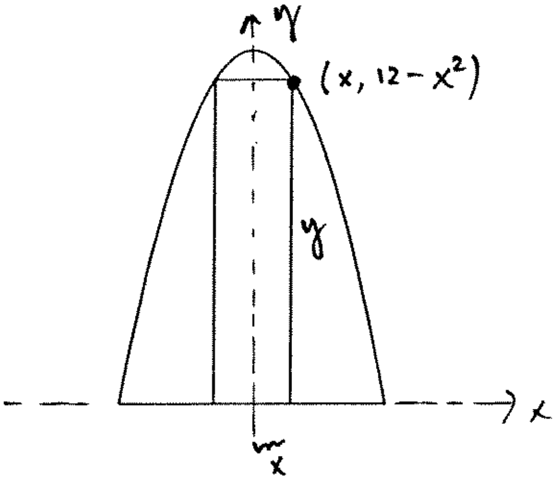
$$3 = 6 + \int_1^2 f(x)\,dx$$

so $\int_1^2 f(x)\,dx = -3$

so $\int_1^2 4f(x)\,dx = 4\int_1^2 f(x)\,dx = \boxed{-12}$

2a) Consider rectangles which have base on the x-axis and upper corners on the parabola $y = 12 - x^2$, such as in the example shown below. Use calculus to find which of these rectangles has the largest area.

(15 points)



maximize $A = (2x)(y)$

onstraint

$y = 12 - x^2$

$A(x) = 2x(12 - x^2)$

$= 24x - 2x^3$

$0 \leq x \leq \sqrt{12}$ since $y \geq 0$

$A'(x) = 24 - 6x^2 = -6(x^2 - 4)$

$A'(x) = 0$ at $x = \pm 2$

$x \geq 0$ so $x = 2$

$y = 12 - 4 = 8$

$x = 2$
$y = 8$
$A = 2 \cdot 2 \cdot 8 = 32$

2b) Use the concepts we've discussed in this class to explain why your answer to part (2a) actually gives the maximum area. You may use an increasing/decreasing, concavity, or critical point explanation, as long as it is mathematically correct. Any of these 3 is correct:

(5 points)

one way:

Domain $0 \leq x \leq \sqrt{12}$ closed interval

$A(x) \geq 0$ on domain

o attains a maximum value > 0.

+ critical pt

endpoint: $A(0) = A(\sqrt{12}) = 0$

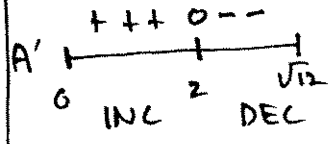
stat pt: $x = 2$ is only one

sing pt: none

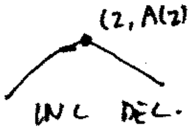
so $A(2)$ must be max value.

another:

$A'(x) = -6(x-2)(x+2)$



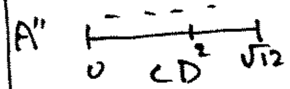
so $A(2)$ is max



another

$A''(x) =$ ~~[illegible]~~ $-12x$

~~[illegible]~~



graph CD on domain, so $A(2)$ is max.

3) Consider the function $f(x) = \frac{x^2}{x-2}$.

natural domain: all $x \neq 2$

3a) Find all asymptotes to the graph $y = \frac{x^2}{x-2}$. Compute the one-sided limits, at either side of the vertical asymptote. (6 points)

$\boxed{x = 2 \text{ vert. asymp.}}$

$$\begin{array}{r|l} & x+2 \\ \hline x-2 & x^2 \\ & x^2 - 2x \\ \hline & 2x \\ & 2x - 4 \\ \hline & 4 \end{array}$$

so $\frac{x^2}{x-2} = x + 2 + \frac{4}{x-2}$

$\lim_{x \to \pm\infty} \frac{4}{x-2} = 0$, so

$\boxed{y = x+2}$ diag. asymp.

$$\boxed{\begin{array}{l} \lim_{x \to 2^+} \frac{x^2}{x-2} = \frac{4}{0^+} = +\infty \\ \lim_{x \to 2^-} \frac{x^2}{x-2} = \frac{4}{0^-} = -\infty \end{array}}$$

3b) Find where $f(x)$ is increasing and decreasing. (6 points)

$$f(x) = \frac{x^2}{x-2}$$

$$f'(x) = \frac{2x(x-2) - x^2 \cdot 1}{(x-2)^2}$$

$$= \frac{2x^2 - 4x - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$$

$f'(x)$: + 0 − DNE − 0 +

0 2 4

INC DEC DEC INC.

$\boxed{\text{i.e. INC on } (-\infty, 0], [4, \infty) \quad \text{DEC on } [0, 2), (2, 4].}$

3c) Find where the graph of $f(x)$ is concave up and concave down. (6 points)

$$f(x) = x + 2 + \frac{4}{x-2}$$

$$f'(x) = 1 - \frac{4}{(x-2)^2}$$

$$f''(x) = -4(-2)(x-2)^{3}(1)$$

$$= +\frac{8}{(x-2)^3}$$

$f''(x)$: − − − DNE + + +

2

CD CU

$\boxed{\text{i.e. } f \text{ CD on } (-\infty, 2) \quad \text{CU on } (2, \infty)}$

3d) Find and classify all stationary points of $f(x)$. (4 points)

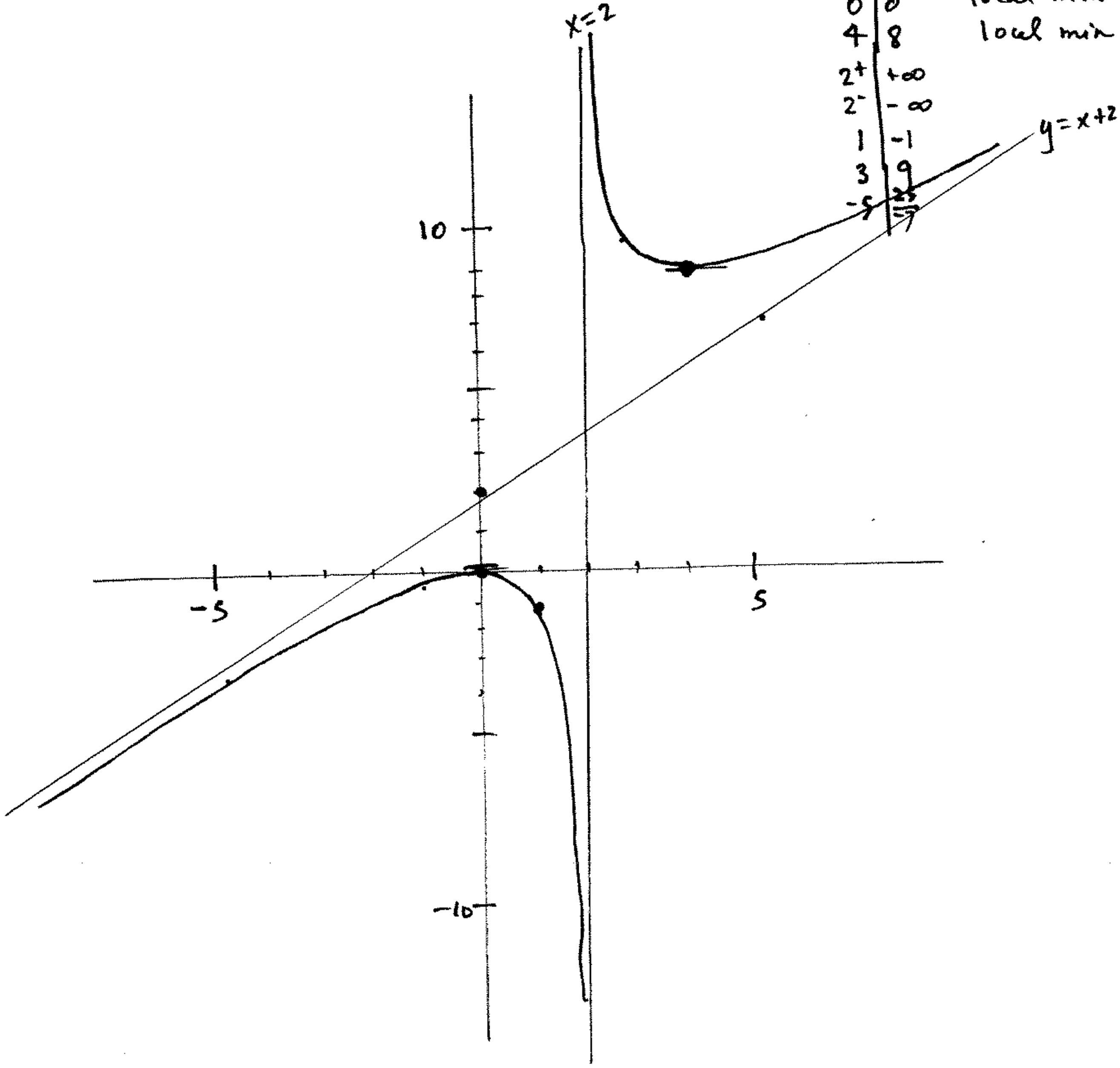
$\boxed{f(0) = 0}$ is local max (by 1st or 2nd deriv test, 3b,c)

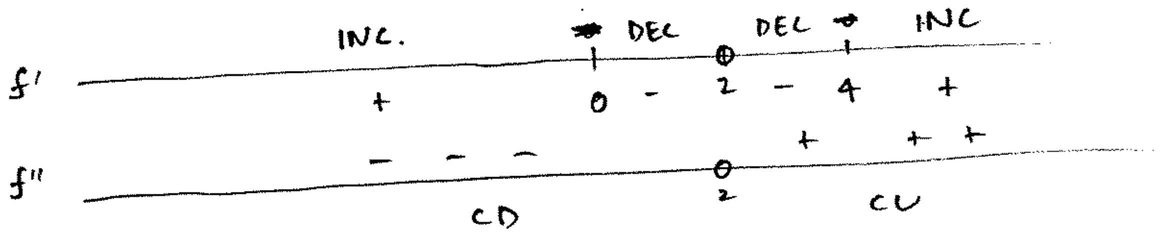
$\boxed{f(4) = \frac{16}{2} = 8}$ is local min (").

3e) Make a careful sketch of the graph of $y = \frac{x^2}{x-2}$, showing all local extrema, asymptotes, and intercepts. The graph should accurately reflect your work on the previous page.

(8 points)

x	y	
0	0	local max
4	8	local min
2^+	$+\infty$	
2^-	$-\infty$	
1	-1	
3	9	
-5	$\frac{25}{-7}$	





4a) Compute $\int_0^2 3x^2\,dx$ using the Fundamental Theorem of Calculus.

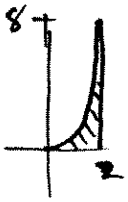
(5 points)

$$= x^3\Big]_0^2 = 8$$

4b) Describe the region in the plane that this definite integral is computing the area of.

(5 points)

the area under $y = 3x^2$, for $0 \leq x \leq 2$
(& above x-axis)



We will now compute $\int_0^2 3x^2\,dx$ directly from the Riemann sum definition, and hopefully get the same answer as in part (4a):

4c) Partition the interval [0,2] into n equal-length subintervals. What is the width of each subinterval, expressed in terms of n?

$$\Delta x = \frac{b-a}{n} = \boxed{\frac{2}{n}}$$

(2 points)

4d) Express the i^{th} subinterval of the partition in terms of the index i and the number n of subintervals.

(3 points)

$$x_i = a + i\,(\Delta x)$$
$$= 0 + \frac{2i}{n}$$
$$= \frac{2i}{n}$$

so ith subinterval $[x_{i-1}, x_i]$

$$\boxed{= \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right]}$$

4e) Use the right-hand endpoints of the subintervals as sample points, and write down the Riemann sum for this definite integral.

(5 points)

$$\bar{x}_i = \frac{2i}{n}$$

$$R_P = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i = \boxed{\sum_{i=1}^{n} 3\left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{24}{n^3}\,i^2}$$

4f) Use the "magic formula" for computing sums of squares to simplify the Riemann sum, and then take the limit as $n \to \infty$. You should get the same answer as in (4a)!

(5 points)

$$\sum_{i=1}^{n} \frac{24}{n^3}\,i^2 = \frac{24}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{24}{n^3}\;\frac{1}{6}\,n(n+1)(2n+1)$$

$$= 4\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$R_P = 4\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$\lim_{n\to\infty} R_P = 4 \cdot 1 \cdot 2 = \boxed{8}$$