Name Solutions

Student I.D. ______________________

Math 1210-3
Exam #2
February 29, 2008

Please show all work for full credit. This exam is closed book, closed note, but you are allowed a scientific calculator. (NOT a graphing calculator, though.) There are 100 points possible, as indicated below and in the exam. Since you only have 50 minutes you should be careful to not spend too long on any one problem. Good Luck!!

| Score | POSSIBLE |
|---|---|
| 1 ______ | 20 |
| 2 ______ | 20 |
| 3 ______ | 25 |
| 4 ______ | 25 |
| 5 ______ | 10 |
| TOTAL ______ | 100 |

1) Compute the following derivatives.

1a) $D_x y$ for $y = 3x^2 + \frac{7}{x^3} + \pi^2$. (6 points)

$7x^{-3}$ a constant

$$\boxed{D_x y = 6x - 21x^{-4} + 0}$$

1b) $f'(t)$ for $f(t) = \frac{\cos(4t^2 - 3)}{t^3 + 1}$. (6 points)

quotient rule

$$\boxed{f'(t) = \frac{(-\sin(4t^2-3))(8t)(t^3+1) - (\cos(4t^2-3))\, 3t^2}{(t^3+1)^2}}$$

or chain rule + product rule

$$f(t) = \cos(4t^2-3) \cdot (t^3+1)^{-1} \quad \text{so} \quad \boxed{f'(t) = -\sin(4t^2-3) \cdot 8t \cdot (t^3+1)^{-1} + \cos(4t^2-3) \cdot (-1)(t^3+1)^{-2}\, 3t^2}$$

1c) $f''(x)$ for $f(x) = \sqrt{1 + \sin(2x)}$. (8 points)

$$f(x) = (1 + \sin 2x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+\sin 2x)^{-1/2} \cdot 2\cos 2x$$

$$\boxed{f''(x) = -\frac{1}{4}(1+\sin 2x)^{-3/2} \cdot 2\cos 2x \cdot 2\cos 2x + \frac{1}{2}(1+\sin 2x)^{-1/2} \cdot 2(-\sin 2x)(2).}$$

2a) Write down the limit definition for the derivative f′(x) of a function f(x).

(3 points)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

2b) Use the limit definition of derivative to compute $D_x \cos(2x)$. Although you should have them memorized, the helpful addition angle formulas will be written on the blackboard.

(12 points)

$$D_x \cos(2x) = \lim_{h\to 0} \frac{\cos(\overbrace{2(x+h)}^{2x+2h}) - \cos 2x}{h}$$

$$= \lim_{h\to 0} \frac{\cos 2x \cos 2h - \sin 2x \sin 2h - \cos 2x}{h}$$

$$= \lim_{h\to 0} \cos 2x \left[\frac{\cos 2h - 1}{h}\right] - \lim_{h\to 0} \sin 2x \frac{\sin 2h}{h}$$

$$= \cos 2x \left(\lim_{h\to 0} \underbrace{\frac{\cos 2h - 1}{2h}}_{\downarrow\, 0} \cdot \underbrace{\frac{2h}{h}}_{\Downarrow\, 2}\right) - \sin 2x \lim_{h\to 0} \underbrace{\frac{\sin 2h}{2h}}_{\downarrow\, 1} \underbrace{\frac{2h}{h}}_{\Downarrow\, 2}$$

$$= 0 - 2\sin 2x \stackrel{!}{=} \boxed{-2\sin 2x}$$

2c) Compute $D_x \cos(2x)$ using derivative rules, to check your work in (2b).

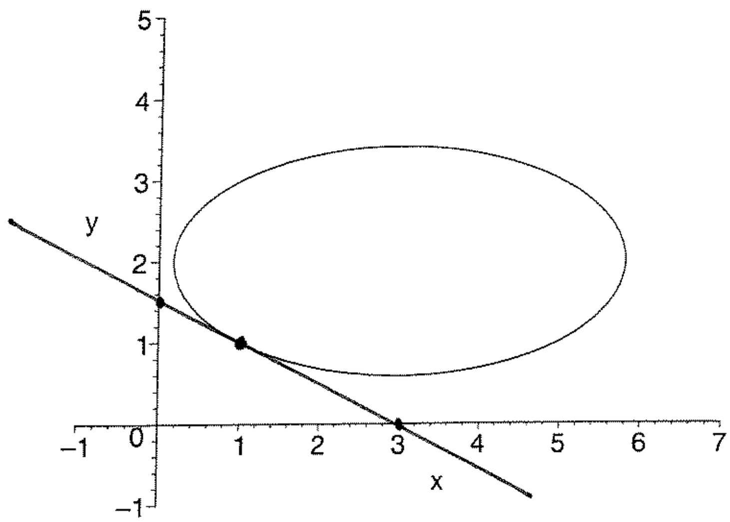
(5 points)

$$D_x \cos 2x = -\sin(2x) \cdot 2$$

chain rule!

corrected in class.

3) The graph of the equation $\frac{(x-3)^2}{8} + \frac{(y-2)^2}{2} = 1$ is an ellipse with center at (3,2). Here's a computer picture:




3a) Use algebra to verify that the point (1,1) is on the graph above.

(5 points)

$$\frac{(1-3)^2}{8} + \frac{(1-2)^2}{2} = \frac{4}{8} + \frac{1}{2} = 1 \checkmark$$

3b) Use implicit differentiation to find the slope of the graph above, at the point (1,1).

(10 points)

$$\frac{1}{8}(x-3)^2 + \frac{1}{2}(y-2)^2 = 1$$

$D_x$: $\frac{1}{8}2(x-3) + \frac{1}{2}2(y-2)y' = 0$

$$\frac{1}{4}(x-3) + (y-2)y' = 0$$

@ (1,1): $\frac{1}{4}(-2) + (-1)y' = 0$ $\quad -\frac{1}{2} - y' = 0$ $\quad \boxed{y' = -\frac{1}{2}}$

3c) What is the equation of the tangent line to the graph above, through the point (1,1)?

(5 points)

$$(y-1) = -\frac{1}{2}(x-1)$$

3d) Sketch the tangent line you just found on to the graph above. Make sure it has the correct x and y-intercepts.

(5 points)

$y = -\frac{1}{2}(x-1) + 1 = -\frac{1}{2}x + \frac{3}{2}$

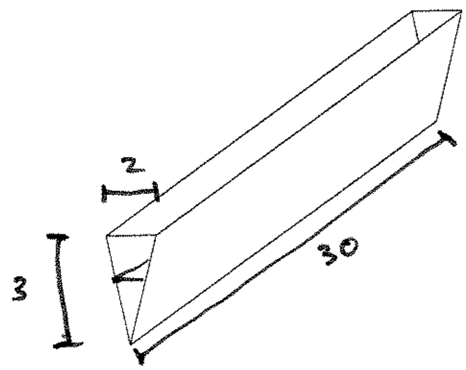
y-intercept $(x=0)$, $y = \frac{3}{2}$

x-intercept $(y=0)$ : $0 = -\frac{1}{2}x + \frac{3}{2}$

$\frac{1}{2}x = \frac{3}{2}$

$\underline{x=3}$

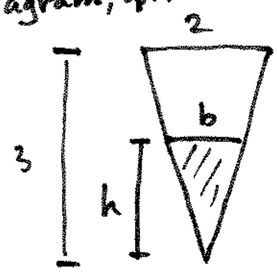
4) Water is slowly filling a water trough, at a constant rate. The trough has an isoceles triangular cross-section, with the "base" at the top of length 2 feet. The triangle's height is 3 feet. The trough is 30 feet long.




When the water is 2 feet deep, you measure that the depth is increasing at a rate of one inch per minute.

4a) How fast is water flowing into the trough? (20 points)

Make, Label diagram, with fns of time




(2) translate English into math

Find $D_t V$ $\quad V$ = volume

when depth $h = 2$ ft

and $h'(t) = \frac{1}{12}$ ft/min

$V = (\text{area of cross-section})\,\text{length}$

$= (\frac{1}{2} bh) 30 \quad ft^3$

$V = 15\,bh$

(3) Relate functions

Similar $\Delta$'s, $\left.\begin{array}{l} \frac{b}{h} = \frac{2}{3} \\ b = \frac{2}{3}h \end{array}\right\}$ so $V = 15\left(\frac{2}{3}h\right)h$

$V = 10\,h^2$

$D_t$: $\quad V'(t) = 20\,hh'$, when $h=2$, $h' = 1/12$

(4) take $D_t$

(5) Plug in values and Solve for unknown rate

$\Rightarrow \boxed{V'(t) = \frac{20 \cdot 2}{12} = \frac{40}{12} = \frac{10}{3}}$ ft³/min.

4b) You need to run an errand for 15 minutes. Will the trough be overflowing when you get back? Answer this question by figuring out when the trough will be completely full.

(5 points)

Volume left to fill is $\underbrace{\frac{1}{2}2 \cdot 3 \cdot 30}_{\text{total capacity}} - \underbrace{\frac{1}{2}2 \cdot \frac{4}{3} \cdot 30}_{\text{already filled}} = 30\left(3 - \frac{4}{3}\right) = \frac{150}{3}\ ft^3$

$= 50\ ft^3$

Since $V'(t) = \frac{10}{3}$ ft³/min it will take $\frac{50}{10/3} = 50 \cdot \frac{3}{10}\ \cancel{ft^3}\ \frac{min}{\cancel{ft^3}}$

$\boxed{= 15 \text{ minutes to fill}}$

answer: tank will be just full when you get back!

5) The cube root of 27 is 3. Use differentials to estimate the cube root of 26. (If you have a scientific calculator you may wish to compare your approximation to the exact value.)

(10 points)

$f(x) = x^{1/3}$

$y = x^{1/3}$

$dy = \frac{1}{3} x^{-2/3}\, dx$

@ $x = 27 \quad y = 3$

$dx = -1$ (displacement to get from 27 to 26)

so $dy = \frac{1}{3}(27)^{-2/3}(-1)$

$= \frac{1}{3}\,\frac{1}{9}(-1) = -\frac{1}{27}$

So $(26)^{1/3} = y + \Delta y$

$\approx y + dy$

$= 3 - \frac{1}{27}$

$= 2\frac{26}{27}$

$\approx 2.963$ whereas calc says

$26^{1/3} = 2.96249\ldots$