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Name SOLUTIONS

Student I.D. ____________________

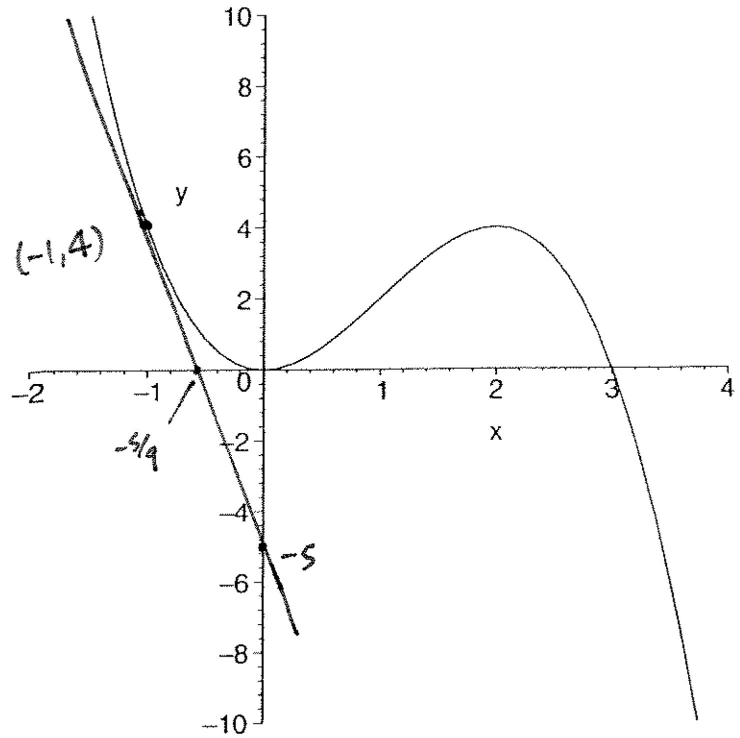
Math 1210-3
Exam #1a
February 1, 2008

Please show all work for full credit. This exam is closed book and closed note. You are allowed a scientfic calculator, but not a graphing calculator. If you aren't sure if your calculator is admissable, ask. There are 100 points possible, as indicated below and in the exam. Since you only have 60 minutes you should be careful to not spend too much time on any one problem. Good Luck!!

Score

	Score	POSSIBLE
1		40
2		10
3		15
4		15
5		20
TOTAL		100

1) Here is a computer sketch of the graph $y = f(x)$, for the polynomial function $f(x) = -x^3 + 3x^2$.



1a) Use polynomial calculus rules to compute the slope of this graph at the point on the graph where $x = -1$. (6 points)

$f(x) = -x^3 + 3x^2$

$f'(x) = -3x^2 + 6x$

$f'(-1) = -3 - 6 = \boxed{-9}$

1b) What is the equation for the line which is tangent to the graph at the point where $x = -1$? (6 points)

point $(-1, f(-1)) = (-1, 4)$

slope $m = -9$

$y - 4 = -9(x+1)$

$y = -9x - 9 + 4$

$\boxed{y = -9x - 5}$

1c) Carefully sketch the tangent line from (1b), into the previous page's picture. First compute the tangent line's x and y-intercepts, label them in your sketch, and then make sure your tangent line sketch passes through them. If you did everything right, the line with your computed slope and intercepts should actually look like the tangent line.

(8 points)

$y = -9x - 5$

$x = 0 \Rightarrow y = -5$ y-intercept

$y = 0 \Rightarrow 9x = -5$ x-intercept

$x = -5/9$

1d) What is the area of the region between the graph $y = f(x)$ and the x-axis, for $0 \le x \le 3$?

(5 points)

$$A = \int_0^3 -x^3 + 3x^2 \, dx$$

$$= -\frac{x^4}{4} + x^3 \Big]_0^3 = \frac{-81}{4} + 27 = -20\tfrac{1}{4} + 27$$

$$\boxed{= 6\tfrac{3}{4}} \quad \left(\frac{27}{4}\right)$$

1e) Suppose a particle is moving with velocity

$$v = -t^3 + 3t^2 \text{ ft/sec}$$

along a number line, for $0 \le t \le 4$ seconds. Suppose the object starts at position $s(0) = 10$ feet. Find the formula for the position function $s(t)$.

(7 points)

$$s(t) = \int -t^3 + 3t^2 \, dt = -\frac{t^4}{4} + t^3 + C$$

$$s(0) = 10 = C$$

$$\boxed{s(t) = -\frac{t^4}{4} + t^3 + 10}$$

1f) What is the greatest value ~~of~~ for $s(t)$, for $0 \le t \le 4$ seconds. (You will end up doing this problem algebraically, but the graph on page 1 may give you a hint.)

(8 points)

turns around when $v(t) = 0$, which is @ $t = 3$ (from graph).

$v(t) = -t^3 + 3t^2 = t^2(-t + 3)$

so $v(t) > 0$ for $t < 3$ (moves in pos. dir.)

$v(t) < 0$ for $t > 3$. (goes back in neg. dir.)

$v(t) = 0$ for $t = 3$ sec

$s(3)$

$= -\frac{81}{4} + 27 + 10 = \boxed{16\tfrac{3}{4} \text{ ft}}$ (see 1d)

2) You computed the derivative of $f(x) = -x^3 + 3x^2$ using derivative rules, in problem (1a). Now use the limit definition of derivative, i.e. expressing the slope of the graph as a limit of secant-line slopes, in order to recompute the derivative $f'(x)$. (Hint: remember the Pascal's triangle expansion $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.)

(10 points)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{-(x+h)^3 + 3(x+h)^2 - [-x^3 + 3x^2]}{h}$$

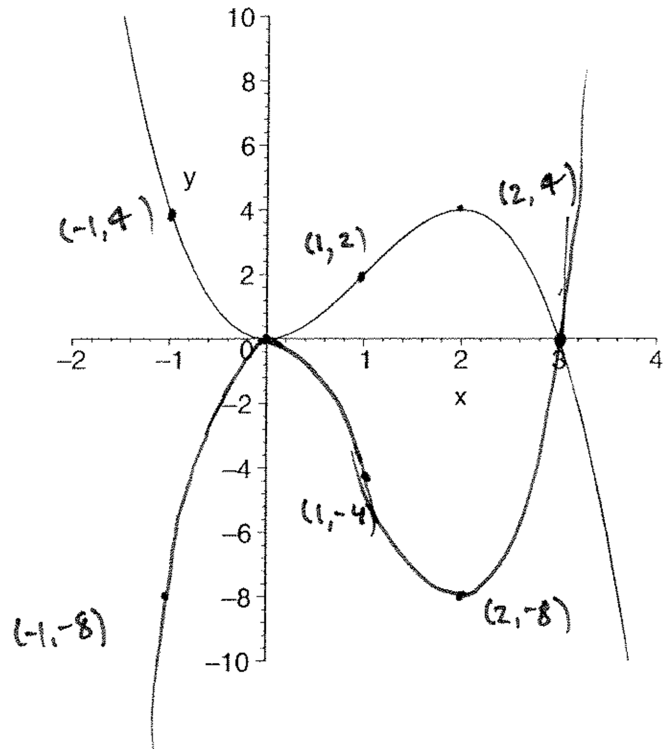
$$= \lim_{h\to 0} \frac{-[\cancel{x^3} + 3x^2h + 3xh^2 + h^3] + 3[\cancel{x^2} + 2xh + h^2] + \cancel{x^3} - \cancel{3x^2}}{h}$$

$$= \lim_{h\to 0} \frac{-3x^2h - 3xh^2 + h^3 + 6xh + 3h^2}{h}$$

$$= \lim_{h\to 0} -3x^2 - \cancel{3xh} + \cancel{h^2} + 6x + \cancel{3h}$$

$$\boxed{f'(x) = -3x^2 + 6x}$$

3) Here, once again is a picture of the graph $y = -x^3 + 3x^2$.



3a) How is the graph of $y = 2x^3 - 6x^2$ related to the graph shown above? Explain in words, and sketch the new graph into the picture above.

(7 points)

$y = -2(-x^3+3x^2)$.

graph is reflected across x-axis and stretched vertically by a factor of 2.

3b) Find an equation whose graph is the reflection of the graph of $y = -x^3 + 3x^2$ across the y-axis. (No need to sketch.)

(4 points)

replace x with −x in the eqn:

$y = -(-x)^3 + 3(-x)^2$

$\boxed{y = x^3 + 3x}$

$x \mapsto x-2$

3c) Find an equation whose graph is the translation of the graph of $y = -x^3 + 3x^2$ by two units to the right, and three units down. (No need to sketch.)

(4 points)

$y = -(x-2)^3 + 3(x-2)^2 - 3$

(or $y+3 = -(x-2)^3 + 3(x-2)^2$)

4) The following problem is related to our discussion of limits and the limit theorems. Suppose we have the estimates

$$|f(x)-6| < 0.02 \text{ and } |g(x)-2| < 0.02 .$$

4a) Explain in words or with inequalities why the estimate for $f(x)$ is the same as saying

$$5.98 < f(x) < 6.02 .$$

(5 points)

$|f(x)-6| < .02$

means

$-.02 < f(x)-6 < .02$

add +6 to each ineq: $5.98 < f(x) < 6.02$

or $|a-b|$ is the distance from a to b on the numberline. So, $|f(x)-6| < .02$ means $f(x)$ is less than .02 from 6, i.e. $5.98 < f(x) < 6.02$

4b) Given the estimates for $f(x)$ and $g(x)$ shown above, we know that $f(x)+g(x)$ must be close to 8. What is the smallest (shortest) interval containing 8 in which you can guarantee $f(x)+g(x)$ must lie?

(5 points)

$5.98 < f(x) < 6.02$

$1.98 < g(x) < 2.02$

add ineq's:

$\boxed{7.96 < f(x)+g(x) < 8.04}$

4c) Given the estimates for $f(x)$ and $g(x)$ shown above, we know that $\dfrac{f(x)}{g(x)}$ must be close to 3. What is the smallest (shortest) interval containing 3 in which you can guarantee $\dfrac{f(x)}{g(x)}$ must lie? (You need not simplify your answer, but if you have a scientific calculator you may wish to check that your answer makes sense.)

(5 points)

in a quotient of positive #'s, it will be the largest when numerator is largest & denom is smallest (and smallest when num is smallest & denom is largest.) Thus

$$\frac{5.98}{2.02} < \frac{f(x)}{g(x)} < \frac{6.02}{1.98}$$

$$\approx \quad 2.96 < \frac{f(x)}{g(x)} < 3.04$$

5) Find the following limits if they exist, or say that they do not exist. Show your reasoning!

5a) $\lim\limits_{y \to (-2)} \dfrac{y^2+5y+6}{y+2}$ $\quad \frac{0}{0}$

(5 points)

$$= \lim_{y \to -2} \frac{\cancel{(y+2)}(y+3)}{\cancel{(y+2)}} = -2+3 = \boxed{1}$$

5b) $\lim\limits_{h \to 0} \dfrac{1}{3} \dfrac{h^3+3h^2+2}{\cos(h)}$ $= \dfrac{1}{3}\left(\dfrac{0+0+2}{1}\right) = \boxed{\dfrac{2}{3}}$

(5 points)

5c) $\lim\limits_{x \to 2} \dfrac{x^3-2x^2+4x-8}{x^2-4}$ $\quad \frac{0}{0}$

(5 points)

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x^2+4)}{\cancel{(x-2)}(x+2)} = \frac{4+4}{4} = \frac{8}{4} = \boxed{2}$$

(we factored the top "by observation", but you can also use long division:

$$\begin{array}{r} x^2+4 \\ x-2 \,\overline{)\, x^3-2x^2+4x-8} \\ \underline{x^3-2x^2} \\ 0+4x-8 \\ \underline{4x-8} \\ 0 \end{array}$$

quotient is x^2+4

5d) $\lim\limits_{h \to 0} \dfrac{(1+h)^{10}-1}{h}$ (Hint: One way to do this problem is to recognize it as a limit definition of derivative problem in disguise, and then use a polynomial calculus derivative rule to deduce the answer.)

(5 points)

This is $f'(1)$ for $f(x) = x^{10}$

but $f'(x) = 10x^9$

$\boxed{f'(1) = 10}$

or $(1+h)^{10} = 1 + 10h + h^2(\text{stuff})$

so $\lim\limits_{h \to 0} \dfrac{(1+h)^{10}-1}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{1}+10h + h^2(\text{stuff}) - \cancel{1}}{h}$

$= \lim\limits_{h \to 0} 10 + h(\text{stuff})$

$= 10$