Math 1210-3
Monday Apr. 7

§ 4.3–4.4 – we still need to discuss why FTC1 & FTC2 are true:

<u>FTC 1</u> $\quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ $\qquad$ (if f continuous on $[a,b]$)

<u>FTC 2</u> $\quad \int_a^b f(x)\,dx = F(b)-F(a)$, $\qquad$ (if $F'=f$ on $[a,b]$, and f cont.)

<u>Remark</u>: We've already seen that FTC2 implies FTC1, since

if $\int_a^x f(t)\,dt = F(x)-F(a)$ $\qquad$ (FTC2)

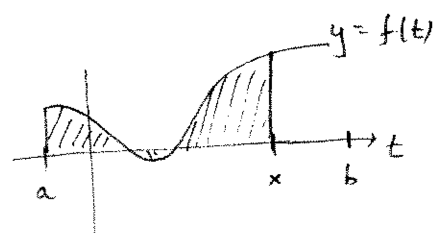
then $D_x \int_a^x f(t)\,dt = F'(x)-0 = f(x)$.

But we haven't actually proven either FTC1 or FTC2.
We'll prove FTC1, and then deduce FTC2 from it.

<u>FTC1</u>: (First Fundamental Theorem of Calculus)

Let $f(x)$ be continuous on $[a,b]$
Define the <u>accumulation function</u>

$$A(x) = \int_a^x f(t)\,dt$$



Then $A'(x) = f(x)$

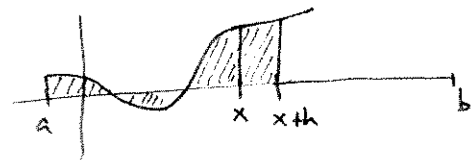
proof: Let's use the limit definition of derivative!

$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}.$$

We will consider $\lim_{h\to 0^+}$; the $\lim_{h\to 0^-}$ is analogous.

For $h>0$, $\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt$



So
$$\lim_{h\to 0^+} \frac{A(x+h)-A(x)}{h} = \lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Let m = minimum value of $f(t)$, $x \le t \le x+h$

Let M = maximum value of $f(t)$, $x \le t \le x+h$.

Thus
$$\underbrace{\int_x^{x+h} m\,dt}_{mh} \le \int_x^{x+h} f(t)\,dt \le \underbrace{\int_x^{x+h} M\,dt}_{Mh}$$

$\div h$:
$$m \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M.$$

Now "squeeze" theorem! Since f is continuous at x, both

m = min value of $f(t)$, $x \le t \le x+h$
M = max value of $f(t)$, $x \le t \le x+h$

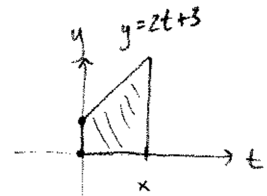
approach $f(x)$ as $h \to 0^+$:
$$f(x) \le \lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x)$$

i.e. $$f(x) \le \lim_{h\to 0^+} \frac{A(x+h)-A(x)}{h} \le f(x).$$

similarly, $$f(x) \le \lim_{h\to 0^-} \frac{A(x+h)-A(x)}{h} \le f(x)$$

i.e. $A'(x) = f(x)$ ■

<u>Exercise 1</u> Verify FTC1 without FTC2, for
$$D_x \int_0^x 2t+3\,dt = 2x+3.$$



Use geometry.

FTC2 follows in one step from FTC1:

FTC2: $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ if $F' = f$ on $[a,b]$ and f is cont.

proof: by FTC1, $A(x) = \int_a^x f(t)\,dt$ is an antideriv of f.

$$A(a) = \int_a^a f(t)\,dt := 0.$$
$$A(b) = \int_a^b f(t)\,dt$$

so $\displaystyle\int_a^b f(x)\,dx = A(b) - A(a).$

But $F(x) = A(x) + C$ since all antiderivs of f differ by a constant.

so $F(b) - F(a) = A(b) + C - (A(a) + C) = A(b) - A(a).$

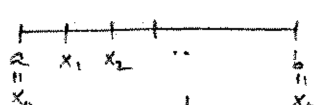
i.e. $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ ■

4.5 Mean value theorem for integrals.
Symmetry time savers

efinition: If f is integrable on the interval $[a,b]$ then the average value of f on $[a,b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

eason for def: Partition $[a,b]$ into n equal length subintervals



$\Delta x = \frac{b-a}{n}$

The average of $f(x_1), f(x_2), \ldots f(x_n)$ is

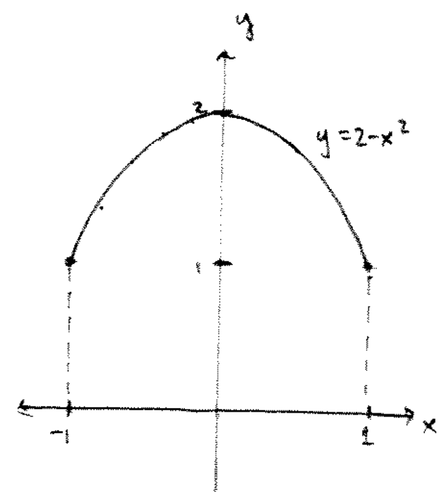
$\displaystyle\frac{1}{n}\sum_{i=1}^{n} f(x_i)$. The "average value of f" should be the limit of this average, as $n \to \infty$.

t,
$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) = \frac{1}{b-a}\underbrace{\sum_{i=1}^{n} f(x_i)\,\Delta x_i}$$

as $n \to \infty$ this converges to $\displaystyle\int_a^b f(x)\,dx$

so the limit of these actual averages is $\displaystyle\frac{1}{b-a}\int_a^b f(x)\,dx.$

Exercise 2 : Find the average value of $f(x) = 2-x^2$ on the interval $[-1,1]$.
Interpret this value as the height of a rectangle with base $-1 \leq x \leq 1$ on the x-axis, where the rectangle has the same area as is under the graph.



Mean Value Theorem for Integrals : If $f(x)$ is continuous on $[a,b]$, then there is at least one c, $a<c<b$, at which $f(c)$ equals the average value of f, i.e.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

proof: This is old MVT in disguise: $\frac{1}{b-a}\int_a^b f(x)\,dx \overset{\text{FTC}}{=} \frac{1}{b-a}\left(F(b)-F(a)\right)$ $\quad (F'=f)$

$= F'(c)$ old MVT, some $a<c<b$.

$= f(c)$

Exercise 3: Verify MVT for Integrals, for Exercise 2 example.

n = 83