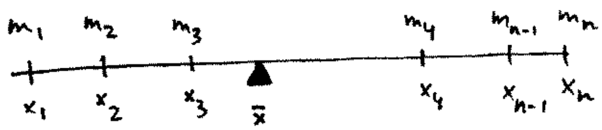
Math 1210-3

Tuesday April 22

Problem Sessions
Thursday JTB 140 9:30-11:00 am
(focus on webworks)
Monday night next week
JTB 140 7-9 pm.
(focus on final exam.)

§5.6 Moments and center of mass

(final section - it will probably take a lot of Wednesday to finish this section, i.e. "today's" notes. There will be a "special" problem session Thurs from 9:30-11:00 am, to help ameliorate our lack of time.)



picture a very crowded teeter totter

The balance point for masses distributed along a line is the $\bar{x}$ which yields a zero "moment":

$$\sum_{i=1}^{n} (x_i - \bar{x}) m_i = 0$$

can be motivated as pivot point at which it takes no net work to rotate the configuration, e.g. working against gravity

$$\sum x_i m_i - \bar{x} \sum m_i = 0$$

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} := \frac{M}{m}}$$

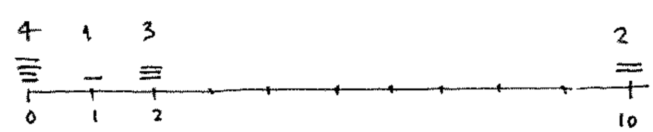
$\leftarrow$ Moment wrt $x=0$.

$\leftarrow$ total mass

Exercise 1 What is the center of mass for masses of

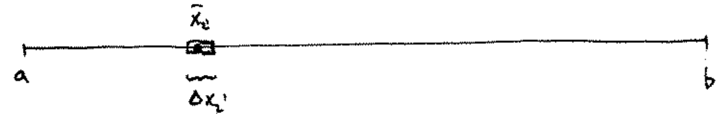
$m_i =$ 4, 1, 3, 2 kg

at $x_i =$ 0, 1, 2, 10 m on a number line?



Add calculus! Consider a rod with varying density (mass/length) $\delta(x)$, $a \le x \le b$

partition,
approximate,
limit



$\Delta m_i \approx$
$\delta(\bar{x}_i)\Delta x_i$

$$\bar{x} \approx \frac{\displaystyle\sum_{i=1}^{n} \bar{x}_i \big(\overbrace{\delta(\bar{x}_i)\Delta x_i}^{\Delta m_i}\big)}{\displaystyle\sum_{i=1}^{n} \underbrace{\delta(\bar{x}_i)\Delta x_i}_{\Delta m_i}}$$

limit as $\|P\| \to 0$

$$\bar{x} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx} := \frac{M}{m}$$

moment (w.r.t. origin)

total mass

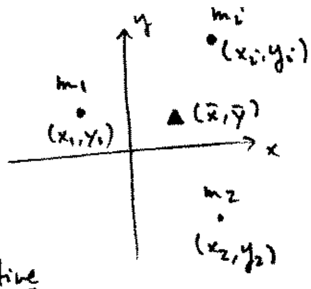
<u>Exercise 2</u> A wire of varying density (and diameter) has density

$$\delta(x) = .004\,x^3 \quad \text{gm/cm}$$
$$0 \le x \le 10 \text{ cm.}$$

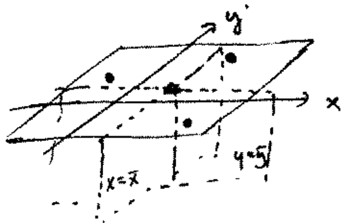
a) Find its total mass

b) Find its center of mass

point masses in the x-y plane :



3-d
<u>perspective</u>



Think of the masses attached at points on a massless horizontal plane
If the points balance on
a vertical plane along $x = \bar{x}$,
and on a vertical plane along
$y = \bar{y}$, then they will balance
at the point $(\bar{x}, \bar{y})$

signed distance to x-axis

$$M_x = \text{moment with respect to x-axis} = \sum_{i=1}^{n} m_i y_i$$

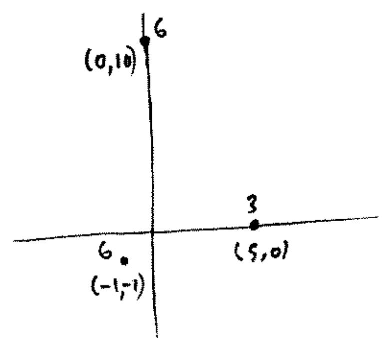
$$M_y = \text{" " " y-axis} = \sum_{i=1}^{n} m_i x_i$$

signed dist to y-axis

$$m = \sum_{i=1}^{n} m_i \quad \text{total mass}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\sum m_i x_i}{\sum m_i}$$ (looks like old formula, this way)

$$\bar{y} = \frac{M_x}{m} = \frac{\sum m_i y_i}{\sum m_i}$$



<u>Exercise 3</u>

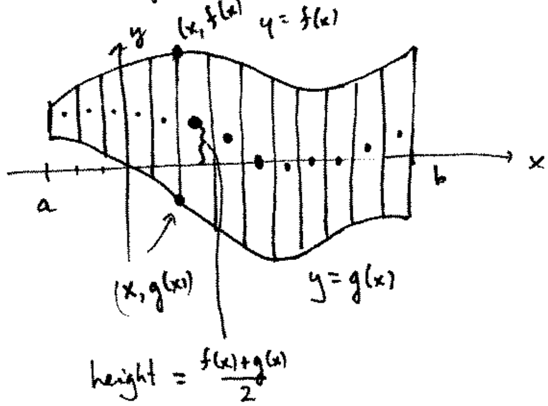
$m_1 = 3$ at $(5,0)$
$m_2 = 1$ at $(0,10)$
$m_3 = 6$ at $(-1,-1)$

Find $(\bar{x}, \bar{y})$

In multivariable Calculus (Math 2210) you learn how to find moments and centers of mass for 2-d & 3-d objects with varying mass-density functions.

Using single variable calc we can handle <u>lamina</u>, (thin planar sheets), with <u>constant</u> density δ mass/area.

Consider a region:



this one's tricky!

ht of ctr of mass of vertical strip

$$\Delta m \approx \delta\,(f(x)-g(x))\,\Delta x \qquad \Delta M_y \approx x(\Delta m) \approx x\,\delta\,(f(x)-g(x))\,\Delta x$$

$$\Delta M_x \approx \frac{(f(x)+g(x))}{2}\,\Delta m = \frac{f(x)+g(x)}{2}\,\delta\,(f(x)-g(x))\Delta x = \frac{\delta}{2}(f^2(x)-g^2(x))\Delta x$$

$$m = \int_a^b \delta\,(f(x)-g(x))\,dx\,, \qquad M_y = \int_a^b \delta\, x\,(f(x)-g(x))\,dx\,,$$

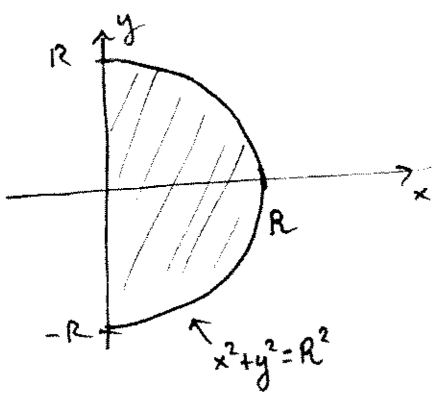
$$M_x = \int_a^b \frac{\delta}{2}\,(f^2(x)-g^2(x))\,dx$$

for $\bar{x}$ & $\bar{y}$ the δ's cancel out:

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x\,(f(x)-g(x))\,dx}{\underbrace{\int_a^b f(x)-g(x)\,dx}_{A}} \leftarrow \text{area!} \qquad \bar{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}(f^2(x)-g^2(x))\,dx}{A}$$

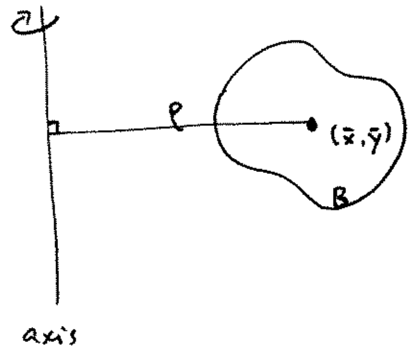
<u>Exercise 4</u> : Find the center of mass of a half-disk lamina:

$(\bar{x}, \bar{y})$ is called the <u>centroid</u>



Cool fact:

Pappus' Theorem (circa A.D. 300 - those amazing Greeks!)



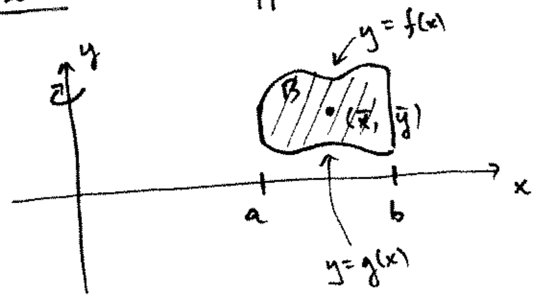
Rotate region B about an axis on one side of it.

Let ρ be the distance from axis to centroid
Let B have area A.
Then the volume of revolution is

$$V = (2\pi\rho)A$$

(like the volume of a straight cylinder of length $2\pi\rho$ & cross-section area A!)

Exercise 5 Prove Pappus' Thm in case:



Use cylindrical shells for the volume.

Exercise 6 What is the volume of this doughnut?:
(rotate circle abt y-axis)