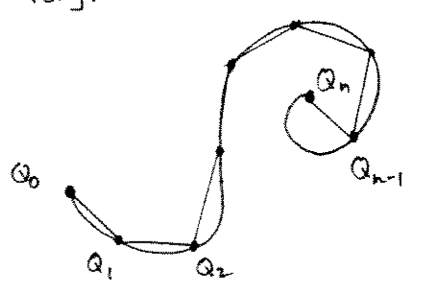
Math 1210-3
Wednesday April 16.

Friday quiz: §5.1-5.3
areas & volumes

§5.4 Lengths of curves

Mathematically (in the absence of string!) the length of a curve is a limit of approximate lengths:



$$\text{Approximate length} = \sum_{i=1}^{n} \text{length}(\overline{Q_{i-1}Q_i})$$

$$\text{Length} = \lim_{\max(\text{length}\,\overline{Q_{i-1}Q_i}) \to 0} \left(\sum_{i=1}^{n} \text{length}(\overline{Q_{i-1}Q_i}) \right)$$

provided the limit exists

As we'll see, this leads to definite integrals, since the sums above will look a lot like certain Riemann sums.

But, what exactly do we mean by a "curve"?

Definition A parametric curve (in the plane), is the collection of points

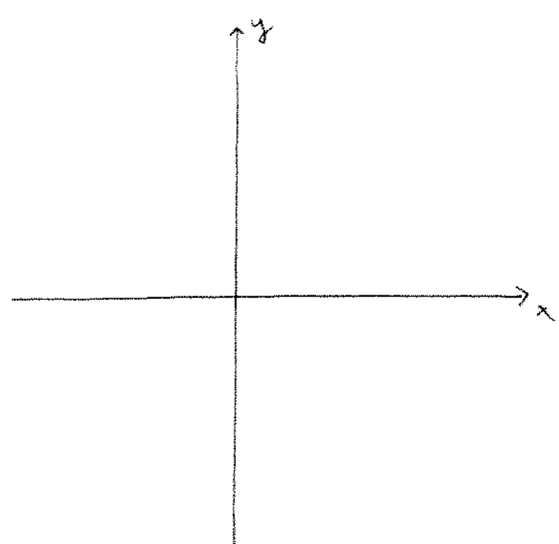
$$x = f(t) \qquad a \le t \le b$$
$$y = g(t)$$

where f and g are continuous functions.

in this case "t" is called a parameter
(often we think of t as time, and think of the (x,y) coordinates as the position at time t, of a particle moving along the curve.)

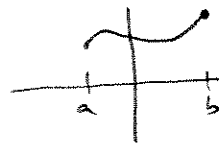
Exercise 1 Describe geometrically, and sketch, the curve

$$x = 3\cos t \quad , \quad 0 \le t \le \tfrac{3}{2}\pi$$
$$y = 3\sin t$$



Some special parametric curves are just graphs:

$x = t$, $y = g(t)$, $a \le t \le b$ $\longrightarrow$ is just the graph $y = g(x)$, $a \le x \le b$



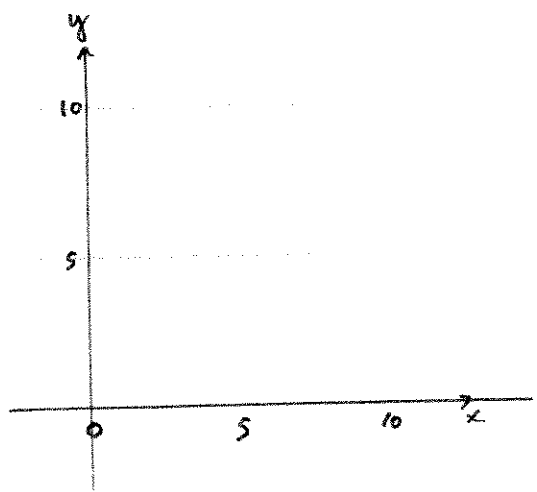
$x = f(t)$, $y = t$, $c \le t \le d$ $\longrightarrow$ is just the graph $x = f(y)$, $c \le y \le d$

As exercise 1 shows, some parametric curves are not graphs with respect to x or y.

Exercise 2a) Sketch the parametric curve

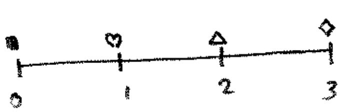
$$x = 2t + 1 \qquad 0 \le t \le 3$$
$$y = t^2 - 1$$

t	(x,y)
0	
1	
2	
3	



Is this curve a graph $y = g(x)$?
or a graph $x = g(y)$?

Exercise 2b) Compute the approximate length of the curve in 2a), as on page 1, using the (x,y) points corresponding to this partition of the t-interval $[0,3]$: Get a decimal value!



<u>Exercise 2c)</u> Here is the computer's approximation the the length of the curve, with $n=30$ equal subdivisions of the t-interval.

```
> f:=t->2*t+1;
  g:=t->t^2-1;
```

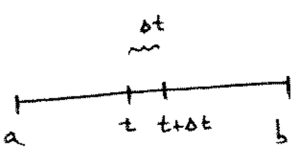
$$f := t \to 2\,t+1$$
$$g := t \to t^2 - 1$$

```
> L:=0.0:  #initialize length approximation
> a:=0:
  b:=3:  #t-interval
  n:=30:  #number of subintervals
> h:=(b-a)/n:  #width of each subinterval
> s:=a:  #initial time
> for i from 1 to n do
      L:=L+evalf(sqrt((f(s+h)-f(s))^2+(g(s+h)-g(s))^2));
          #add the length of the next line segment
      s:=s+h:  #update the time parameter
  od:
  L;   #exhibit final length approximation
```

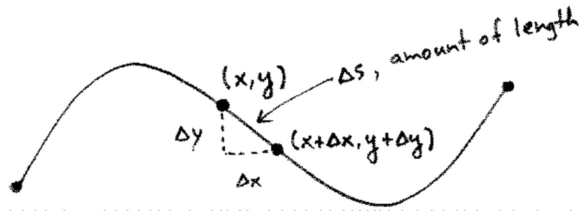
$$11.30448887$$

How to get the exact length, as a definite integral:

$$x=f(t) \qquad x+\Delta x = f(t+\Delta t)$$
$$y=g(t) \qquad y+\Delta y = g(t+\Delta t)$$



parameter interval.



$$\Delta s \approx \text{hypotenuse}$$
$$= \sqrt{(\Delta x)^2+(\Delta y)^2}$$
$$= \begin{cases} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t & x=f(t),\ y=g(t) \\ \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x & x=x,\ y=g(x) \qquad a\le x\le b \\ \sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2+1}\ \Delta y & x=f(y),\ y=y \qquad c\le y\le d \end{cases}$$

for a partition of $[a,b]$,

$$L \approx \sum_{i=1}^{n} \Delta s_i$$

$$L = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \Delta s_i$$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt}\ ,$$

with analogous formulas for graphs

Exercise 3 : Explain, using length formula on page 3, why the exact length of the Exercise 2 curve is

$$\int_0^3 2\sqrt{1+t^2}\,dt$$

We don't have an antideriv for that integrand yet,
but here is the computer's answer... compare to page 3 output:

```
> Int(2*sqrt(1+t^2),t=0..3);
  evalf(%);
```

$$\int_0^3 2\sqrt{1+t^2}\,dt$$ ← definite integral

$$11.30527944$$ ← decimal value

```
> int(sqrt(1+t^2),t);
```

$$\frac{t\sqrt{1+t^2}}{2}+\frac{1}{2}\operatorname{arcsinh}(t)$$ ← antiderivative

Exercise 4 : Verify that the length formula gives correct answer for the $\frac{3}{4}$-circle of radius 3,

$x = 3\cos t$
$y = 3\sin t$ $\quad 0 \le t \le \frac{3}{2}\pi$ $\qquad$ i.e. the Exercise 1 example.