INTRODUCTION TO POLYNOMIAL CALCULUS

by

Joseph L. Taylor

Department of Mathematics

University of Utah

December 2007

INTRODUCTION TO POLYNOMIAL CALCULUS

1. Straight Lines

Given two distinct points in the plane, there is exactly one straight line that contains them both. This is one of the important principles of Plane Geometry. If the plane is equipped with a Cartesian coordinate system, it should be possible to write down an equation for such a line in terms of the x and y coordinates. In this section we shall show how to do this. The notion of slope will be very useful in this task.

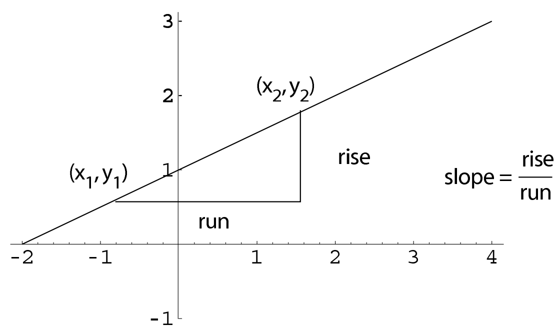


Figure 1.1

The slope of a line is defined to be the change in the y coordinate (the rise) divided by the change in the x coordinate (the run) as we move from one point on the line to a second point on the line (see Figure 1.1). That is, if (x_1, y_1) and (x_2, y_2) are two points on the line, then the slope of the line is the number m, where

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This number does not depend on which two points on the line are chosen. In fact, if two other points (x_3, y_3) and (x_4, y_4) are chosen, then it follows from the similar triangles in Figure 1.2 that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

and so the points (x_3, y_3) and (x_4, y_4) give the same value for the slope m as do the points (x_1, y_1) and (x_2, y_2).

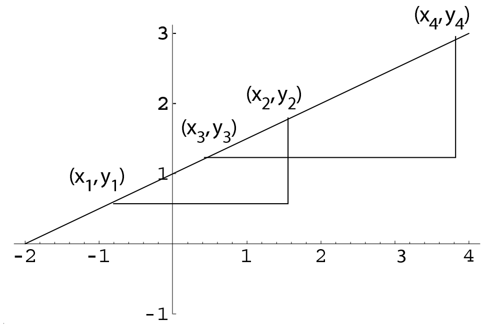


Figure 1.2

The slope measures whether a line rises or falls as we move to the right and how steeply it does so. Positive slope means the line rises to the right, while negative slope means it falls to the right. Slope zero means the line is horizontal, since it means that the y coordinate does not change at all from point to point on the line. A vertical line has no slope (some would say it has infinite slope). This is because the x-coordinates of points on such a line are all the same and so the denominator is zero in the equation defining slope.

Example 1. Find the slope of the line which contains the points $(1, 2)$ and $(3, 5)$.
Solution: Here the rise is $5 - 2 = 3$ and the run is $3 - 1 = 2$ and so the slope is $\frac{\text{rise}}{\text{run}} = 3/2$.

The point-slope form of the equation of a line.

Using slope we can easily write down an equation for any line that is not vertical. Let m be the slope of the line and let (x_0, y_0) be some fixed point on the line. If (x, y) is a variable point on the line, then

$$m = \frac{y - y_0}{x - x_0}$$

and so

$$y - y_0 = m(x - x_0)$$

This is called the *point-slope* form of the equation of a straight line. It gives the equation of the line in terms of the slope m of the line and a point (x_0, y_0) on the line.

Example 2. Find the equation of the line which passes through the point $(1, 3)$ and has slope 2.
Solution: Here the slope m is 2 and the point (x_0, y_0) is $(1, 3)$ and so the equation of the line is

$$y - 3 = 2(x - 1) \quad \text{or} \quad y = 2x + 1$$

Example 3. Find the equation of the line which contains the points $(-1, 2)$ and $(0, 5)$

Solution: The slope of this line is

$$m = \frac{5-2}{0-(-1)} = 3$$

and the line contains the point $(0, 5)$; so the equation of the line is

$$y - 5 = 3(x - 0) \quad \text{or} \quad y = 3x + 5$$

The slope-intercept form of the equation of a line.

If a line crosses the y-axis at the point $(0, b)$, then the number b is called the y-intercept of the line. If the line has slope m, then the point-slope form of its equation (using the point $(0, b)$) is $y - b = m(x - 0)$ or

$$y = mx + b$$

This is the *point-intercept* form of the equation of the line. Note that every line except a vertical line does cross the y-axis and, thus, has a slope-intercept form for its equation.

Example 4. What is the equation of the line with slope -5 and y-intercept 4 ?
Solution:

$$y = -5x + 4$$

The general equation of a line.

Every equation of the form

$$Ax + By + C = 0$$

is the equation of a line and, conversely, every line has an equation that can be put in this form. A non-vertical line has an equation of the form $y = mx + b$, which can be written

$$mx - y + b = 0$$

which is of the form $Ax + By + C$ with $A = m$, $B = -1$ and $C = b$. On the other hand, a vertical line has the form

$$x = k$$

where k is a constant. This can be re-written as

$$x - k = 0$$

which is of the form $Ax + By + C = 0$ with $A = 1$, $B = 0$, and $C = -k$.

Example 5. Find the slope and y-intercept for the line which is described by the equation $2x + 3y - 6 = 0$.

Solution: If we solve for y in this equation, we get the equation

$$y = -\frac{2}{3}x + 2$$

The equation of the line is now in slope-intercept form and we can just read off the slope $(-\frac{2}{3})$ and the y-intercept (2).

Parallel and perpendicular lines.

If the two lines in Figure 1.3 are parallel, then the two triangles are similar, from which it follows that the two lines have the same slope. On the other hand, if the two lines have the same slope then the two triangles in Figure 1.3 must be similar, since they contain right angles with proportional adjacent sides. This implies that the two lines are parallel since they make the same angle with a horizontal line. Thus, **two lines are parallel if and only if they have the same slope.**

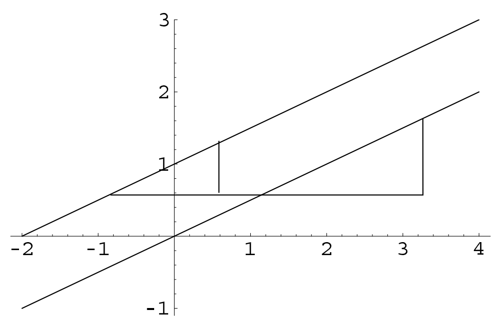


Figure 1.3

What about perpendicular lines? There is an old carpenter's trick for checking that an angle is a right angle. One measures off three units along one leg and makes a mark and then measures four units along the other leg and makes another mark. The angle is then a right angle if and only if the distance between the two marks is five. This works because $3^2 + 4^2 = 5^2$ and an angle in a triangle is a right angle if and only if

$$a^2 + b^2 = c^2$$

where a and b are the lengths of the two adjacent sides of the angle and c is the length of the opposite side (the Pythagorean Theorem).

Let us see how we can apply this to tell when two lines are perpendicular. We may as well assume that neither line is horizontal, since it is easy to see when a line is perpendicular to a horizontal line (it must be vertical). Then the first line intersects the x-axis in a point $(x_1, 0)$ and the second line intersects the x-axis in a point $(x_2, 0)$. Let's suppose the two lines are not parallel and so they intersect each other in a point (x_0, y_0) (see Figure 1.4). Then they intersect at a right angle if and only if $a^2 + b^2 = c^2$, where a is the distance from (x_0, y_0) to $(x_1, 0)$, b is the distance from (x_0, y_0) to $(x_2, 0)$ and c is the distance from $(x_1, 0)$ to $(x_2, 0)$. This leads to the equation

$$(x_1 - x_0)^2 + y_0^2 + (x_2 - x_0)^2 + y_0^2 = (x_2 - x_1)^2$$

which simplifies to

$$2x_0^2 + 2y_0^2 - 2x_0x_1 - 2x_0x_2 = -2x_1x_2$$

On dividing by 2 and rearranging terms this becomes

$$y_0^2 = -x_0^2 + x_0x_1 + x_0x_2 - x_1x_2$$

or, after factoring,

$$y_0^2 = -(x_0 - x_2)(x_0 - x_1)$$

or

$$\frac{y_0}{x_0 - x_2} = -\frac{x_0 - x_1}{y_0}$$

which says that

$$m_2 = -\frac{1}{m_1}$$

where m_1 is the slope of the first line and m_2 is the slope of the second line. Thus, **two non horizontal lines are perpendicular if and only if the slope of the second is the negative reciprocal of the slope of the first.**

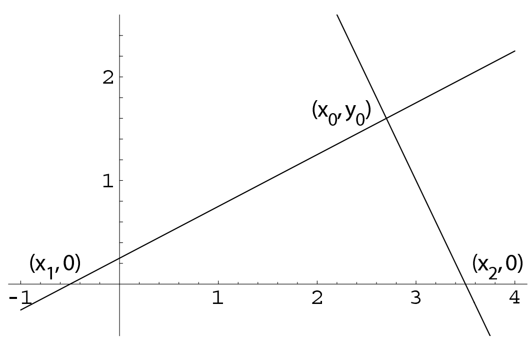


Figure 1.4

Example 6. Find the equation of the line which is parallel to the line with equation $2x + 3y = 5$ and passes through the point $(1, 2)$.

Solution: We solve for y in the equation of the first line in order to put its equation in the form

$$y = -\frac{2}{3}x + \frac{5}{3}$$

This shows that the line has slope $-\frac{2}{3}$. The line which has the same slope and passes through the point $(1,2)$ has the equation

$$y-2=-\frac{2}{3}(x-1) \quad \text{or} \quad y=-\frac{2}{3}x+\frac{8}{3}x \quad \text{or} \quad 2x+3y=8$$

Example 7. Find the equation of the line which is perpendicular to the line $2y+x=3$ and meets it at the point $(1,1)$.

Solution: If we solve the first equation for y it becomes

$$y=-\frac{1}{2}x+\frac{3}{2}$$

which tells us that its slope is $-\frac{1}{2}$. A line perpendicular to this line will, therefore, have slope 2. Since the line we seek must pass through the point $(1,1)$, its equation is

$$y-1=2(x-1) \quad \text{or} \quad y=2x-1$$

Problem Set 1

In problems (1) through (6) find the slope of the line containing the indicated two points:

(1) $(0,1)$ and $(1,2)$;
(2) $(2,3)$ and $(4,7)$;
(3) $(1,1)$ and $(3,2)$;
(4) $(1,4)$ and $(3,2)$;
(5) $(-2,3)$ and $(3,1)$;
(6) $(-2,0)$ and $(0,2)$.

In problems (7) through (12) find the equation of the line with the indicated slope and passing through the indicated point:

(7) slope 2 and point $(0,0)$;
(8) slope 5 and point $(1,2)$;
(9) slope -3 and point $(2,-1)$;
(10) slope $\frac{1}{2}$ and point $(1,1)$;
(11) slope $-\frac{2}{3}$ and point $(0,5)$;
(12) slope 7 and point $(-2,0)$.
(13) Find the equation of the line with slope 3 and y-intercept 1.
(14) Find the equation of the line with slope $\frac{4}{3}$ and y-intercept 2.
(15) Find the slope and y -intercept for the line with equation $6x-2y=4$.
(16) Find the slope and y -intercept for the line with equation $2x+5y=3$.
(17) Find the equation of the line which passes through $(1,1)$ and is parallel to the line $y=3x+2$.

(18) Find the equation of the line which passes through $(2, -1)$ and is parallel to the line which passes through $(2, 0)$ and $(3, 2)$.
(19) Find the equation of the line which passes through $(1, 0)$ and is perpendicular to the line $y = 3x + 2$.
(20) Find the equation of the line which bisects the line segment from $(0, 0)$ to $(2, 4)$ at a right angle.
(21) Find the equation of the line which passes through $(0, 1)$ and is perpendicular to the line $x = 3$.
(22) Find the equation of the line which passes through $(2, 0)$ and is perpendicular to the line $y = 1$.
(23) If a perpendicular line is drawn from the point $(1, 1)$ to the line $2y - x = 4$, at what point does it meet this line? What is the distance from the point $(1, 1)$ to the line $2y - x = 4$.
(24) What is the distance from the point $(0, 1)$ to the line $y = 2x - 3$?
(25) What is the distance from the line $y = 2x$ to the parallel line $y = 2x + 3$?

2. Slope of a Curve

Based on the previous section, we know about the slope of a straight line. What about the slope of a curve that is not a straight line? Does this make sense? If so, how do we calculate it?

Let's look at an example, say the curve $y = x^2$. The graph of this curve (Figure 2.1a) makes it clear that, if its slope makes sense, it cannot be a fixed number. To the left of $x = 0$ the curve slopes downward (negative slope) while to the right of $x = 0$ the curve slopes upward (positive slope) and it rises more steeply the further to the right we go. Thus, if it makes sense at all, the slope must depend on where we are on the curve.

In fact, the slope of the curve $y = x^2$ does make sense at each point (x, y) on the curve (but it changes as the point changes). This is suggested by the fact that if a microscopically tiny piece of the curve is magnified enough to be visible, then it looks like a straight line. This is illustrated by the graphs in Figure 2.1 which show the curve near the point $(1, 1)$ magnified by various factors. These graphs suggest that as we take smaller and smaller segments of the curve, centered at $(1, 1)$, and magnify them, they look more and more like segments of a straight line. The slope of this straight line should be what we mean by the slope of the curve at the point $(1, 1)$.

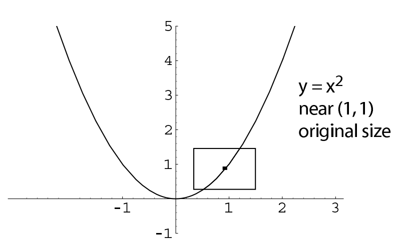


Figure 2.1 a

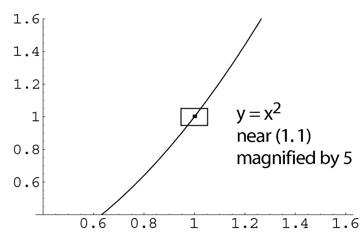


Figure 2.1b

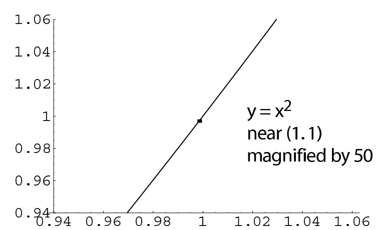


Figure 2.1 c

How can we calculate this slope? We do the same thing we would do if the curve were a straight line. We calculate the change in y divided by the change in x as we move from one point on the curve to another. However, now we choose special points. We let the first one be $(1, 1)$ itself and we choose the second one to be near $(1, 1)$. The nearer to $(1, 1)$ we make it, the more the curve between these two points looks like a straight line and the closer our ratio will be to the slope of this line. Let's try this for some choices of points on the curve $y = x^2$. In each case we will be calculating the rise divided by the run between a first point, $(1, 1)$, and some nearby second point on the curve. In other words we will be calculating the slope of the line joining these two points.

With second point $(3, 9)$ we get

$$\text{slope} = \frac{9-1}{3-1} = \frac{8}{2} = 4$$

With second point $(2, 4)$ we get

$$\text{slope} = \frac{4-1}{2-1} = \frac{3}{1} = 3$$

With second point $(1.5, 2.25)$ we get

$$\text{slope} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{.5} = 2.5$$

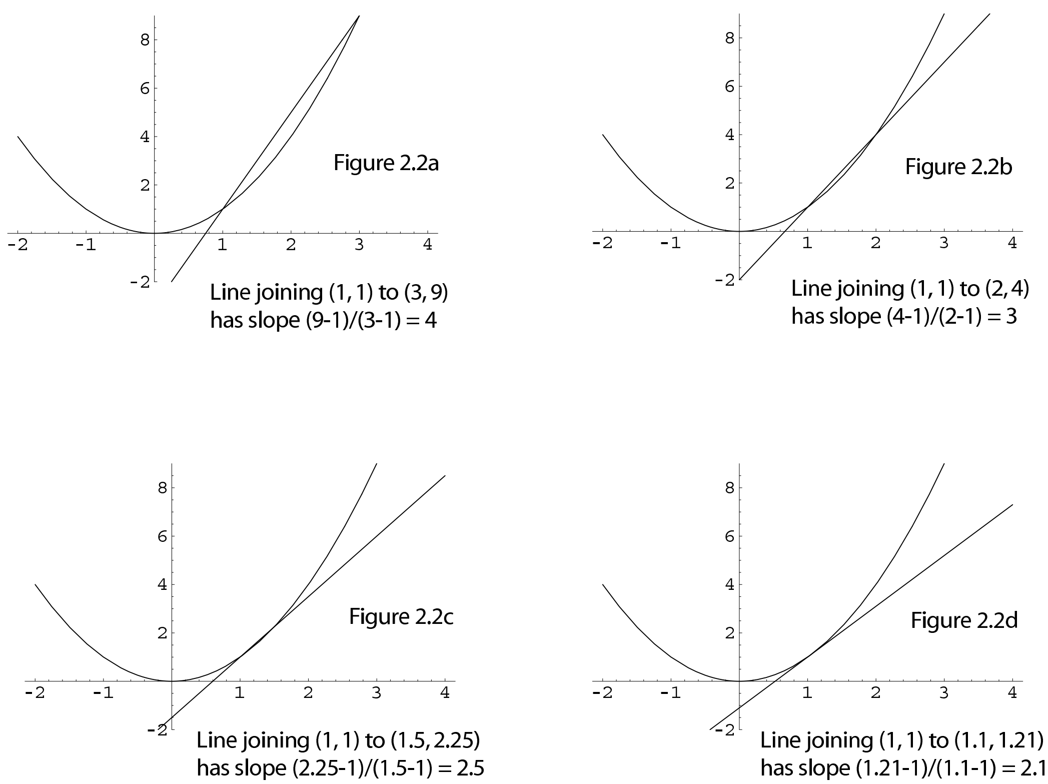
With second point $(1.1, 1.21)$ we get

$$\text{slope} = \frac{1.21 - 1}{1.1 - 1} = \frac{.21}{.1} = 2.1$$

With second point $(1.01, 1.0201)$ we get

$$\text{slope} = \frac{1.0201 - 1}{1.01 - 1} = \frac{.0201}{.01} = 2.01$$

A graphic description of what is happening here is shown in Figure 2.2.



One can now guess that if we continue doing this for second points that are closer and closer to $(1, 1)$, the slope we are calculating will simply get closer and closer to the number 2. This would mean that the slope of the curve $y = x^2$ at the point $(1, 1)$ is 2.

Let us now take a perfectly general second point – one that is obtained by changing the x coordinate of the first point by an amount h so that the second point becomes

$(1+h,(1+h)^2)$. The smaller h is chosen, the closer the second point is to $(1,1)$. The slope we get for the line joining $(1,1)$ to this second point is then

$$\text{slope} = \frac{(1+h)^2-1}{h} = \frac{2h+h^2}{h} = 2+h$$

Now if h is chosen very small then this number will be very close to 2. In other words, as h approaches 0 our slope approaches 2. This should convince us that the slope of the curve $y=x^2$ at the point $(1,1)$ is 2.

The fact that $2+h$ approaches 2 as h appoaches 0 is commonly written

$$\lim_{h\to 0}(2+h)=2.$$

Here $\lim\limits_{h\to 0}(2+h)$ is shorthand for "the limit of $2+h$ as h approaches 0" and it simply means the number that $2+h$ approaches as h approaches 0.

The method used above works for other curves as well.

Example 1. Find the slope of the curve $y=x^2-3x$ at the point where $x=2$.
Solution: For the function $f(x)=x^2-3x$ we have

$$f(2)=-2$$
$$f(2+h)=(2+h)^2-3(2+h)=-2+h+h^2$$

Thus, the slope of the curve when $x=2$ is

$$\text{slope} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{h+h^2}{h} = \lim_{h\to 0}(1+h)=1$$

Returning to the curve $y=x^2$, we would like a formula for the slope of this curve at any point of the curve, not just at the point $(1,1)$. We use the same technique. Given a point (x,x^2) on the curve, we move to a nearby second point $(x+h,(x+h)^2)$, obtained by changing the x coordinate of the first point by an amount h. Then we calculate the slope of the line joining these two points. It is

$$\text{slope} = \frac{(x+h)^2-x^2}{h} = \frac{2xh+h^2}{h} = 2x+h$$

As h approaches zero this slope approaches the number $2x$. In other words

$$\lim_{h\to 0}(2x+h)=2x$$

We conclude that the slope of the curve $y=x^2$ at the point (x,x^2) is $2x$.

The Derivative

The preceding discussion leads to the following definition of the slope of a curve which is the graph of a function f. This, in turn, leads to the definitions of the derivative of a function.

Definition A. *The graph of a function f is said to have slope m at a point $(x, f(x))$ on the graph provided*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = m$$

That is, provided the expression $\frac{f(x+h)-f(x)}{h}$ approaches m as h approaches 0. In this case, we call the number m the derivative of f at x and denote it by $f'(x)$.

Thus, the derivative of a function f is another function f' of the same variable x. Its value at x is the slope of the graph of f at the point $(x, f(x))$. In other words, it is the instantaneous rate of change of y with respect to x as we pass through the point $(x, f(x))$ while moving along the graph of f.

For now we will take the statement "$\frac{f(x+h)-f(x)}{h}$ approaches m as h approaches 0" as intuitively understood, but later in the course we will need to make this idea more precise. We will do this when we study limits. As we shall see in the next section, when f is a polynomial it is quite easy to see what happens to the expression $\frac{f(x+h)-f(x)}{h}$ as h approaches 0 and so to study derivatives of polynomials we do not need a sophisticated study of limits.

The discussion preceding the above definition shows that the derivative of x^2 is $2x$. We give two other examples of the calculation of a derivative.

Example 2. Find the derivative of a constant function $f(x) = c$.
Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

Thus, the derivative of a constant function is 0. This just reflects the fact that the graph of a constant function is a horizontal line and, thus, has slope 0.

Example 3. Find the derivative of the function $f(x) = 3x - 2$.
Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h) - 2 - (3x - 2)}{h} = \lim_{h \to 0} 3 = 3.$$

Thus, the derivative of the function $f(x) = 3x - 2$ is the constant 3. This is not surprising, since the graph of the function $f(x) = 3x - 2$ is a straight line with slope 3.

Tangent Line to a Curve

A line is tangent to a curve at a point if the line and the curve both pass through the point and have the same slope at the point. More precisely, if $y = f(x)$ is the equation of a curve, then a point on the curve has the form $(a, f(a))$ for some real number a. If it exists, the slope of the curve at this point is $m = f'(a)$. The line which has this same slope and also passes through the point $(a, f(a))$ is called the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. The point–slope form of the equation of this line is

$$y - f(a) = f'(a)(x - a).$$

We emphasize that the graph of a function f has a tangent line at one of its points $(a, f(a))$ if and only if the function f has a derivative at a. The slope of the tangent line is this derivative $f'(a)$. Note, a is fixed and $f'(a)$ is a number, not a function of x.

Example 4. Find an equation of the tangent line to the curve $y = x^2$ at the point $(1, 1)$.

Solution: We determined earlier that the slope of the curve $y = x^2$ at $(1, 1)$ is 2. This is the derivative, $2x$, of x^2 evaluated at $x = 1$. The line which has this slope and passes through $(1, 1)$ has equation $y - 1 = 2(x - 1)$ in point–slope form. If we were to graph this line and the curve $y = x^2$ on the same graph, the result would be indistinguishable from Figure 2.2d.

Problem Set 2

In problems 1 - 8 you are to find the slope of the curve $y = f(x)$ at the point where x has the indicated value by calculating $\frac{f(x+h)-f(x)}{h}$ and determining what number it approaches as h approaches 0:

(1) $f(x) = 3x + 2$, x = 1;
(2) $f(x) = x^2$, $x = 0$;
(3) $f(x) = x^2$, $x = 2$;
(4) $f(x) = x^2 - 3$, $x = 1$;
(5) $f(x) = x^2 + 2x - 1$, $x = 0$;
(6) $f(x) = 3x^2 - 2$, $x = 1$;
(7) $f(x) = x^3$, $x = 1$;
(8) $f(x) = x^3$, $x = 0$.

In problems 9 - 14 you are to find $f'(x)$ by calculating $\frac{f(x+h)-f(x)}{h}$ and determining what it approaches as h approaches 0:

(9) $f(x) = x$;
(10) $f(x) = 2x + 5$;
(11) $f(x) = 3x^2$;
(12) $f(x) = x^2 - 2x + 3$;
(13) $f(x) = x^3$;
(14) $f(x) = x^3 + x^2$.

In the next two problems find an equation for the tangent line to the given curve at the given point (you may use any information derived in the text or examples of this section):

(15) the curve $y = x^2$ and the point $(-2, 4)$;
(16) the curve $y = x^2 - 3x$ and the point $(2, -2)$.

3. Derivative of a Polynomial

In the last section we defined the derivative $f'(x)$ of a function $f(x)$ to be the slope of the curve $y = f(x)$ at the point $(x, f(x))$. This, in turn, is defined to be the number that the expression $\frac{f(x+h)-f(x)}{h}$ approaches as h approaches 0. In other words

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

We will now use this definition to calculate the derivative of any polynomial. We begin by calculating the derivative of the monomial x^n for each value of n. For $n = 0, 1, 2, 3$ this was done in the previous section and its problem set. However, we will repeat these calculations here in order to demonstrate the pattern that emerges.

If $f(x) = x^0 = 1$ then

$$f'(x) = \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0} 0 = 0$$

If $f(x) = x^1 = x$ then

$$f'(x) = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} 1 = 1$$

If $f(x) = x^2$ then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} (2x + h) = 2x$$

If $f(x) = x^3$ then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2$$

The pattern here suggests that, for any natural number n, the derivative of x^n should be nx^{n-1}. This is, in fact, true. The proof is not difficult. From the definition of derivative, we have that

$$(x^n)' = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

If we expand $(x+h)^n$ using the binomial theorem we obtain

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n$$

It is not important here to know exactly each term in this expansion. It is important to know the first two terms and the fact that every term except the first two has a factor of

h raised to a power of at least two. When we subtact x^n from this expansion and divide by h we obtain

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}$$

Here, every term except the first one has a factor of h. Thus, when we take the limit as h appoaches 0, all terms except the first one will vanish. We conclude that

$$(x^n)' = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}$$

This proves the following theorem:

Theorem A. *If n is any non-negative integer, then*

$$(x^n)' = nx^{n-1}$$

We now know the derivatives of a large number of functions. For example:

$$(x^4)' = 4x^3$$
$$(x^{10})' = 10x^9$$
$$(x^{95})' = 95x^{94}$$

Next we would like to be able to find the derivative of a polynomial like $x^3 + 4x^2 - x + 5$ which is a linear combination of monomials. We need the following theorem:

Theorem B. *If f and g are functions and c is a constant, then*

$$(cf)' = cf' \tag{1}$$
$$(f+g)' = f' + g' \tag{2}$$

That is, the derivative of a constant times a function is that constant times the derivative of the function and the derivative of the sum of two functions is the sum of their derivatives.

We won't prove this theorem now. Its proof is in the textbook and will be done later in the course. Using this theorem and the preceding one, we can now find the derivative of any polynomial.

Example 1. Find the derivative of $3x^2 + 2x + 5$.
Solution:

$$\begin{aligned}(3x^2 + 2x + 5)' &= (3x^2)' + (2x)' + (5)' && \text{Th B(2)}\\ &= 3(x^2)' + 2(x)' + (5)' && \text{Th B(1)}\\ &= 3 \cdot 2x + 2 \cdot 1 + 0 && \text{Th A}\\ &= 6x + 2\end{aligned}$$

Example 2. Find the derivative of $x^5 - 11x^3 + 9x + 2$.
Solution:

$$\begin{aligned}(x^5 - 11x^3 + 9x + 2)' &= (x^5)' + (-11x^3)' + (9x)' + (2)' \\ &= 5x^4 - 11 \cdot 3x^2 + 9 \cdot 1 + 0 \\ &= 5x^4 - 33x^2 + 9\end{aligned}$$

Example 3. Find the slope of the curve $y = x^4 - 3x^2 + 2$ at the point $(1, 0)$.
Solution: We first find the derivative of $x^4 - 3x^2 + 2$:

$$(x^4 - 3x^2 + 2)' = 4x^3 - 3 \cdot 2x + 0 = 4x^3 - 6x$$

We then evaluate the derivative at $x = 1$ to get the slope of the curve at the point $(1, 0)$. Thus, the answer is

$$4 - 6 = -2$$

Example 4. Find the rate of change of the function $f(x) = x^3 - 4x$ with respect to x when $x = 2$.
Solution: We first find the derivative of $f(x)$:

$$f'(x) = (x^3 - 4x)' = 3x^2 - 4$$

We then evaluate the derivative at $x = 2$ to find the rate of change of $f(x)$ with respect to x at $x = 2$. Thus, the answer is

$$3 \cdot 2^2 - 4 = 8$$

Example 5. Find an equation of the tangent line to the curve $y = x^3 - 2x^2 + 1$ at the point $(2, 1)$. **Solution:** The derivative of $x^3 - 2x^2 + 1$ is $3x^2 - 4x$. At $x = 2$ this is 4. Thus, the curve $y = x^3 - 2x^2 + 1$ has slope 4 at the point $(2, 1)$. The point–slope form of the equation of the tangent line at this point is thus $y - 1 = 4(x - 2)$.

Velocity and Acceleration

Much of Physics and Engineering is concerned with the mathematics of moving bodies. Suppose a body moves along a straight line. If we fix an origin for this line and units for measuring distance on the line, then the position of the body at any time t is described by its coordinate on the line, often denoted by $s(t)$. The velocity $v(t)$ of the body is then defined to be the rate of change of $s(t)$ with respect to t - that is, the derivative $s'(t)$ of

$s(t)$. Similarly, the acceleration $a(t)$ of the body is defined to be the rate of change of $v(t)$ with respect to t - that is, the derivative $v'(t)$ of $v(t)$. In summary,

$$v(t) = s'(t)$$
$$a(t) = v'(t)$$

Example 5. If a ball if dropped off the top of a 64 foot high building, then its height $s(t)$ above the ground t seconds later is described by the formula

$$s(t) = -16t^2 + 64$$

What is its velocity when it hits the ground? What is its acceleration at any time?

Solution: The ball hits the ground when

$$-16t^2 + 64 = 0.$$

This happens when $t^2 = 4$; that is, when $t = 2$. Thus, we need to know the velocity of the ball when $t = 2$. But

$$v(t) = s'(t) = -16 \cdot 2t + 0 = -32t$$

At time $t = 2$ this is -64. Thus, the ball hits the ground with velocity -64 ft/sec.

The acceleration at any time t is

$$a(t) = v'(t) = (-32t)' = -32$$

Thus, the acceleration is a constant -32 ft/sec/sec.

Problem Set 3

(1) Find the derivative of x^9.
(2) Find the derivative of $2x^{50}$.
(3) Find the derivative of $3x - 6$.
(4) Find the derivative of $x^3 - 2x + 4$.
(5) Find the derivative of $2x^4 + x^3 - 5x^2 + x + 2$.
(6) Find the derivative of $x^{11} - 2x^9 + 15x$.
(7) Find the slope of the curve $y = x^3$ at the point $(1, 1)$.
(8) Find the slope of the curve $y = x^2$ at the point $(0, 0)$.
(9) Find the slope of the curve $y = x^3 - x^2$ at the point $(1, 0)$.
(10) Find an equation for the tangent line to the curve $y = x^4 - 2x^3 + 5x - 3$ at $(2, 7)$.
(11) Find an equation for the tangent line to the curve $y = x^{10} - x^5$ at $(1, 0)$.
(12) For what values of x does the curve $y = x^2 - 2x + 3$ have positive slope? Negative slope? Zero slope?
(13) If a ball is thrown straight up in such a way that its height t seconds later is

$$s(t) = -16t^2 + 32t + 6$$

find the velocity of the ball at t seconds after it is thrown. At what time t does the ball reach its maximum height (hint: the velocity will be positive before this time and negative after it). How high does the ball get?

(14) In the previous problem, what is the acceleration of the ball at any time t?

4. Antiderivatives of Polynomials

We have introduced the operation of differentiation (finding the derivative of a function) and shown how to differentiate any polynomial. In this section we will study the reverse operation.

Definition A. *If $f(x)$ is a function, then an antiderivative for f is a function having $f(x)$ as its derivative.*

Antidifferentiation undoes or reverses differentiation. If f is the derivative of g, then g is an antiderivative of f. Notice that we said "an antiderivative", not "the antiderivative". This is because a function which has an antiderivative always has infinitely many. In fact, if g is an antiderivative of f then $g + c$ is also an antiderivative of f for any constant c. This follows from the fact that the derivative of a constant is zero.

Given a function f, does f have an antiderivative? This is a difficult question in general and will be studied in some detail later in the course. However, for polynomials it is quite easy to see that the answer is yes. We begin by looking at the monomial x^n, where n is a non-negative integer. Is this the derivative of some other polynomial? If we differentiate x^{n+1} we get

$$(x^{n+1})' = (n+1)x^n$$

If we divide by $n + 1$ this leads to the equation

$$\left(\frac{x^{n+1}}{n+1}\right)' = x^n$$

Thus, we have found an antiderivative for x^n, namely $\frac{x^{n+1}}{n+1}$. This is not the only antiderivative for x^n, since we can add any constant to a function without changing its derivative. Thus, any function of the form

$$\frac{x^{n+1}}{n+1} + c$$

where c is a constant, will have derivative x^n.

Definition B. *If $f(x)$ is a function, the set of all antiderivatives for $f(x)$ is denoted*

$$\int f(x)\,dx$$

and is called the indefinite integral of $f(x)$.

We have shown that every function of the form $\frac{x^{n+1}}{n+1} + c$ is an antiderivative for x^n. Conversely, every antiderivative for x^n is of this form (as long as we consider the domain for x^n to be the whole real line or, at least, an interval on the real line). However, to prove this requires one of the big theorems of calculus - the Mean Value Theorem - which will not be discussed until later in the course. For now we will simply assume that this is true. Then we can state the following theorem:

Theorem A. *If n is a non-negative integer, then*

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

where c ranges over all constants.

Remember, this theorem simply says that the set of antiderivatives for x^n is the set of functions of the form $\frac{x^{n+1}}{n+1} + c$. If we apply this for $n = 0, 1, 2$ we get:

$$\int 1\,dx = \int x^0\,dx = x + c$$

$$\int x\,dx = \int x^1\,dx = \frac{x^2}{2} + c$$

$$\int x^2\,dx = \frac{x^3}{3} + c$$

If h is an antiderivative for f and a is a constant, then

$$(ah)' = ah' = af$$

and so ah is an antiderivative for af. Similarly, if h is an antiderivative for f and k is an antiderivative for g, then

$$(h+k)' = h' + k' = f + g$$

and so $h + k$ is an antiderivative for $f + g$. Thus, we have proved the following theorem

Theorem B. *If f and g are functions and a is a constant, then*

$$\int af(x)\,dx = a\int f(x)\,dx \tag{1}$$

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{2}$$

In other words, the integral of a constant times a function is the constant times the integral of the function and the integral of the sum of two functions is the sum of the integrals of the two functions.

Theorems A and B combined allow us to integrate (find the indefinite integral of) any polynomial.

Example 1. Integrate $x^3 + 3x - 6$.

Solution:

$$\begin{aligned}\int (x^3+3x-6)\,dx &= \int x^3\,dx + \int 3x\,dx + \int -6\,dx && \text{Th B(2)}\\ &= \int x^3\,dx + 3\int x\,dx - 6\int 1\,dx && \text{Th B(1)}\\ &= \frac{x^4}{4} + \frac{3x^2}{2} - 6x + c && \text{Th A}\end{aligned}$$

Note that we can always check our answer to an integration problem by differentiating it to see if we get back the original function. For example, let's differentiate the answer in the preceding example:

$$\left(\frac{x^4}{4} + \frac{3x^2}{2} - 6x + c\right)' = \frac{4x^3}{4} + \frac{3\cdot 2x}{2} - 6 + 0 = x^3 + 3x - 6$$

Indeed, we do get back our original function and this verifies that our answer was correct.

Example 2. Find the antiderivative of $x^3 - 3x^2$ that has value 0 when $x = 1$.
Solution: All antiderivatives of $x^3 - 3x^2$ are given by

$$\int (x^3 - 3x^2)\,dx = \frac{x^4}{4} - x^3 + c$$

When $x = 1$ this becomes

$$1/4 - 1 + c = -3/4 + c$$

Thus, if we want the antiderivative that has value 0 when $x = 1$, we should choose $c = 3/4$. The answer is then

$$\frac{x^4}{4} - x^3 + \frac{3}{4}$$

Example 3. The acceleration experienced by an object due to the force of gravity is -32ft/sec^2. A projectile is fired straight up with an initial velocity of 128 ft/sec, after which the only force acting on it is gravity. What is its velocity t seconds later? When does it reach its maximum height?
Solution: Acceleration is the rate of change of velocity with respect to time - that is, it is the derivative of velocity as a function of time. We know the acceleration is -32ft/sec^2. Thus, the velocity $v(t)$ is an antiderivative of the constant function -32. So

$$v(t) = \int -32\,dt = -32t + c$$

When $t = 0$ the velocity is the initial velocity 128 ft/sec. Thus, the constant c must be 128. Therefore, the velocity at time t is

$$v(t) = -32t + 128$$

The object reaches its maximum height when its velocity drops to zero (it is not going up anymore). This happens when

$$-32t + 128 = 0$$

that is, when $t = 4$ seconds.

Example 4. For the preceding example, assume that the object was fired from an initial height of 10 feet off the ground and then find the height above the ground of the object at any time t. Find the maximal height achieved by the object.

Solution: Velocity is the rate of change of distance (in this case height) with respect to time. That is, it is the derivative of height with respect to time. Thus, the height $s(t)$ we are seeking is an antiderivative of of the velocity $v(t) = -32t + 128$. This means

$$s(t) = \int (-32t + 128)\, dt = -16t^2 + 128t + c$$

When $t = 0$ the height is the initial height 10 feet. Thus, $c = 10$ and

$$s(t) = -16t^2 + 128t + 10$$

The maximum height occurs when $t = 4$ by the previous example. So the maximal height achieved is $-16 \cdot 16 + 128 \cdot 4 + 10 = 266$ feet.

Problem Set 4

(1) Find $\int (2x - 3)\, dx$.
(2) Find $\int (3x^2 - 4x + 5)\, dx$.
(3) Find $\int (x^5 + 2x^3 + 1)\, dx$.
(4) Find $\int (10x^9 - 8x)\, dx$.
(5) Find the antiderivative of $x^2 - 5$ that has value 2 when $x = 0$.
(6) Find the antiderivative of $8x^3 - 2x$ that has the value 4 when $x = 1$.
(7) Find the antiderivative of $2x^3$ that has the value 1 when $x = 1$.
(8) Find the antiderivative of $x^3 - x$ that has the value 1 when $x = 2$.
(9) If a ball is thrown straight up with initial velocity of 64 ft/sec, what will its velocity be after t seconds? At what time t will it achieve its maximum height?
(10) If the ball in the last problem was thrown from an initial height of 6 feet, what will its height be after t seconds? What is the maximum height it achieves?

5. Definite integrals

In this section we take up an issue which, at first, seems completely unrelated to what has gone before. This is the problem of calculating the area of a curved figure. We know how to calculate the area of a rectangle with sides of lengths a and b – it is just the product ab. Other simple rectilinear figures such as triangles and figures that can be cut up into rectangles or triangles can easily be figured out based on our knowledge of the area of a rectangle. However, how does one calculate the area of a figure with curved sides? Does this even make sense?

The areas of some curved figures (such as circles) were known to the ancient greek philosophers. The method that they used to calculate areas of of such figures is the following: the figure was closely approximated by a polygonal figure (a figure made up of a finite number of triangles overlapping only along edges). The areas of these triangles where then added up. The result was an approximation to the area of the original figure. The better the polygonal approximation to the original figure the better the approximation to its area. This method was used to calculate the area of a circle to a high degree of accuracy.

In modern day Calculus we use this same method. We will use it to both define the area of certain types of figures and to calculate the area. This is a big subject and in these notes will give just a brief introduction to the ideas involved. A more detailed exposition of this topic occurs in the textbook and will be covered later.

The regions we will consider in this course can all be described as follows: think of cutting the plane into pieces by cutting along finitely many "nice" curves which may cross each other and then taking one of the pieces so produced. The "nice" curves are curves which are given by equations of the form $y = f(x)$ or $x = g(y)$, where f and g are "nice" functions. What do we mean by a nice function? We mean a function which is continuous in the following sense.

Definition A. *A function f is said to be continuous at a if $\lim_{x\to a} f(x) = f(a)$. If f is continuous at every point of an interval $[a, b]$ then it is said to be continuous on $[a, b]$.*

Thus, a function is continuous at a if it has a value $f(a)$ at the point a, and if the limit of $f(x)$ as x approaches a also exists, and the two are equal.

Area Under a Curve

Suppose that $y = f(x)$ is a nonnegative continuous function on the interval $[a, b]$. The region bounded above by the curve $y = f(x)$, below by the x-axis, on the left by the vertical line $x = a$ and on the right by the vertical line $x = b$ is a region of the type described above. For short, we will call this the region under the curve $y = f(x)$ between a and b. We will describe a procedure for defining a calculating the area of such a figure. This is a short preview of the definition that will appear later in the texbook.

Suppose that f is a positive continuous function on the interval $[a, b]$. Let us subdivide the interval by a sequence of partition points $\{x_i\}$:

$$x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Consider the sum

$$f(x_1)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \cdots + f(x_n)(x_n - x_{n-1})$$

which can be written, using the summation notation

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}). \tag{1}$$

This is the sum of the areas of the rectangles bounded by the vertical lines $x = x_{i-1}$, $x = x_i$, the x-axis and the horizontal line $y = f(x_i)$ (see Figure 5.1). Thus it represents the area of a polygonal region approximating the region under the curve $y = f(x)$ between a and b. If we divide the figure more finely, using more points $\{x_i\}$, which divide the interval $[a, b]$ into smaller and smaller subintervals, these sums approach a limit which is then defined to be the area of the region under the curve $y = f(x)$ between a and b.

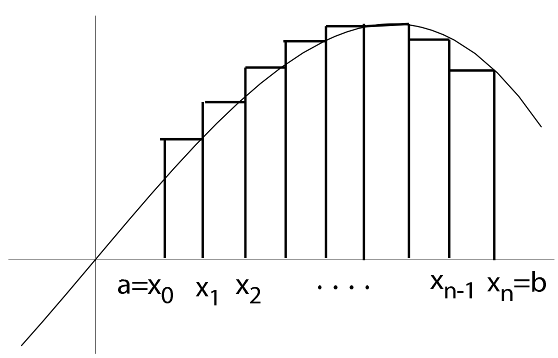


Figure 5.1

It turns out that the sum (2) makes sense even if the function f is not positive on $[a, b]$ and, as long as f is still continuous, it still approaches a limit as the partition becomes finer and finer. The number it approaches is no longer the area under a curve, but it is still a number related to area. It turns out to be the sum of the areas under those parts of the curve which lie above the x-axis minus the sum of the areas above the curve and below the x-axis for those pieces of the curve which dip below the x-axis. In any case, this number is called the Riemann integral (or definite integral) of f between a and b and is denoted

$$\int_a^b f(x)dx.$$

A fact that will be important in what follows is this: the definite integral is a number. It depends on the function f and the endpoints a and b of the interval $[a, b]$, but it does not depend on x or even on the name we give to this variable. If we decide to call our independent variable t instead, then the integral would be written as

$$\int_a^b f(t)dt.$$

This is the same number as before. Changing the name of the variable of integration x to t has no effect on the integral.

The Fundamental Theorem of Calculus

The preceding discussion of area and Riemann integrals seems to have nothing to do with our previous work on derivatives and antiderivates and what we called the indefinite integral. However, there is a close connection and it is this connection that makes calculus such a powerful tool. The connection is spelled out in the Fundamental Theorem of Calculus.

The following is one form of the Fundamental Theorem of Calculus, which will be proved later in the course. Here we will just give a brief indication of how the proof goes.

Theorem A. *If f is a continuous function on the interval $[a, b]$, and F is any antiderivative of f, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Indication of Proof. For simplicity, we give our indication of proof only in the case where f is a positive function.

We wish to use x for something else, so we will change our variable of integration to t. For x any point between a and b , let

$$A(x) = \int_a^x f(t)\,dt.$$

Since f is a positive function, this is the area under the curve $y = f(t)$ between $t = a$ and $t = x$. Let's calculate $A'(x)$. For a small positive increment h, the difference $A(x+h)-A(h)$ is the area of the region under the curve between x and $x + h$ (see Figure 5.2). If we compare the area of this region with the area of its largest inscribed rectangle and its smallest circumscribed rectangle, we conclude that

$$mh \leq A(x + h) - A(h) \leq Mh$$

where M is the maximum value that f has on the interval $[x, x + h]$ and m is the minimal value that f has on this interval. On dividing by h, we have

$$m \leq \frac{A(x + h) - A(h)}{h} \leq M. \tag{2}$$

A similar argument comparing areas works in the case when h is a small negative increment and leads to the same inequalities (2). As $h \to 0$ it is reasonable to assume that m and M will both approach $f(x)$ and this is, in fact, true provided f is continuous at x. We conclude

$$\lim_{x \to 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

Thus $A(x)$ is an antiderivative of f.

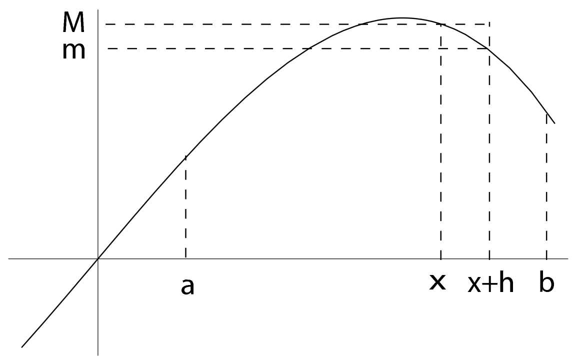


Figure 5.2

If F is any other antiderivative,

$$A(x) - F(x) = C,$$

where C is some constant which we can easily find since $A(a) = 0$. Substituting in (1) we obtain $C = -F(a)$. Thus, the area under the curve from a to b is $A(b) = F(b) - F(a)$. That is, $\int_a^b f(t)\, dt = F(a) - F(b)$, as was to be shown.

Why is this only an indication of proof and not a complete proof? Because we made assumptions about the existence of a maximum and minimum value for a continuous function f on an interval $[x, x+h]$ and we assumed they must both approach $f(x)$ as $h \to 0$. We also assumed a number of properties of the area concept defined above. These issues will be addressed later in the textbook.

Example 1. Find the area under $y = x^n$ from 0 to 1.

Solution: An antiderivative for x^n is $F(x) = \dfrac{x^{n+1}}{n+1}$, so the area is $F(1) - F(0) = \dfrac{1}{n+1}$.

Example 2. Find the area under the curve $y = x^2 + 2x^3$ from 2 to 4.

A notation that is often used in calculus for $F(b) - F(a)$ is $F(x)\big|_a^b$. Thus, $x^3\big|_1^2 = 2^3 - 1^3 = 7$.

Solution:

$$\int_2^4 (x^2 + 2x^3)dx = (\frac{1}{3}x^3 + \frac{2}{4}x^4)\Big|_2^4$$
$$= \frac{1}{3}64 + \frac{1}{2}256 - \frac{1}{3}8 - \frac{1}{2}16 = \frac{416}{3}.$$

The Fundamental Theorem of Calculus (Theorem A) holds for non-positive functions as well. It gives us a simple way to calculate the definite integral of a function f if we can find an antiderivative for f. That is, if f is a continuous function on an interval $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is any antiderivative of f.

Example 3.. Suppose a particle moves in a straight horizontal line so that its velocity directed to the right at time t is $v(t) = t^2 - t^3$ meters per minute. How far to the left or right of the initial position of the particle will it be after 2 minutes?

Solution: The position $s(t)$ of the particle at time t is an antiderivative for $v(t) = t^2 - t^3$ and so

$$s(2) - s(0) = \int_0^2 (t^2 - t^3)dt.$$

However, $g(t) = \frac{1}{3}t^3 - \frac{1}{4}t^4$ is also an antiderivative for $t^2 - t^3$ and so

$$\int_0^2 (t^2 - t^3)dt = g(2) - g(0) = \frac{8}{3} - \frac{16}{4} - 0 = -\frac{4}{3}:$$

the particle is 4/3 of a meter to the left of its initial position.

Problem Set 5

(1) Find $\int_1^5 (x^2 - 2x + 1)dx$.
(2) Find $\int_0^2 (x^3 + 2)dx$.
(3) Find $\int_0^1 (x^4 - x^5)dx$.
(4) Find $\int_0^1 (x^n - x^{n+1})dx$, for any $n \geq 0$.
(5) Find the area under the curve $y = 3x^2 + 2x + 1$ between $x = 1$ and $x = 2$.
(6) Find the area under the curve $y = x^2 + 5x$ between $x = 3$ and $x = 4$.
(7) Find the definite integral of $y = x^{10} - x^9$ from $x = 1$ to $x = 3$.
(8) A particle travels along a horizontal line so that its velocity at time t is $v(t) = 2t + 3t^2 + 1$ feet per second. Suppose that at time $t = 1$ the particle is at the origin. What is the location of the particle at time $t = 3$?