Now that you’ve been convinced that “rates of change” are everywhere...

12.2.2.2.3 : Differentiability; sum, product, quotient rules.

$f$ is differentiable at $x = c$ means

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$ exists.

Sometimes we substitute $x = c + h$, $h = x - c \to 0$, i.e. $x \to c$:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

**Exercise 1**  Decide from the graph below whether $f'(c)$ exists at the indicated $x$-values - use the geometric meaning of the limit definition of derivative. If the derivative exists, estimate its value.

![Graph of $y = f(x)$ with points labeled at $x = -1, 0, 2, 3, 4$ and $f'(-1), f'(0), f'(2), f'(3), f'(4)$]
Relationship between continuity and differentiability (use Exercise 1 to visualize this)

(a) If \( f(x) \) is differentiable at \( x=c \), then \( f \) is continuous at \( x=c \).

(b) If \( f(x) \) is continuous at \( x=c \), it may or may not be differentiable there.

proof of (a): let \( f(x) \) be differentiable at \( x=c \).

Write

\[ f(x) = f(c) + (f(x) - f(c)) \]

\[ f(x) = f(c) + \left( \frac{f(x) - f(c)}{x-c} \right)(x-c) \]

so \( \lim_{x \to c} f(x) = \)

Exercise 2 Find a function in our "zoo" which shows (b), i.e. continuity does not imply differentiability.
Exercise 3 Use the limit definition of derivative to find $f'(x)$, for $f(x) = \frac{2}{x+7}$

Life is easier if you know...

**Differentiation rules theorem**

1. $(f + g)' = f' + g'$ **sum rule**
2. $(cf)' = cf'$ **constant multiple rule**
3. $(fg)' = fg' + gf'$ **product rule**
4. $(\frac{f}{g})' = \frac{fg' - gf'}{g^2}$ **quotient rule**

(there's a rule for differentiating compositions too, it's called the chain rule, §2.5)

Exercise 4 redo exercise 3 with the quotient rule

Exercise 5 if $n$ is a positive integer, use the quotient rule to find $f'(x)$ for $f(x) = x^{-n} = \frac{1}{x^n}$

Exercise 6 What is $[(x^3 + 6x)(2x^2 + 5)]'$ ? Check ans. by multiplying out first.

Exercise 7 Find the derivative of $f(x) = \frac{\sin x}{\cos x}$. 
Exercise 8  The product rule and quotient rule seem mysterious, but we can prove them!

(a) Show, using the limit definition, that
\[(fg)' = f'g + fg'

(b) If we have time, try \((f/g)'\)! (If not, see page 112 of text).