Tues Sept 11

§1.2 Rigorous limits:

Define what $\lim_{x \to c} f(x) = L$ precisely,

and develop the tools to compute limits and make estimates

warm-up example: The careful study of limits is very much related to the reasoning you use in error analysis for scientific experiments

Consider an experiment to measure g, the acceleration of gravity:

height $= 10 \pm .01$ m

h

object dropped.

time it takes to fall is $1.4 \pm .1$ sec

$s''(t) = -g$

$s(0) = h$

$s'(0) = 0$

$$\boxed{\begin{array}{l} h = 10 \pm .01 \text{ m} \\ t_1 = 1.4 \pm .1 \text{ sec} \end{array}}$$

$$\Rightarrow s'(t) = -gt$$

$$\Rightarrow s(t) = -\tfrac{1}{2} g t^2 + h$$

$$s(t_1) = 0 = -\tfrac{1}{2} g t_1^2 + h$$

$$h = \tfrac{1}{2} g t_1^2$$

$$g = \frac{2h}{t_1^2} = \frac{2(10 \pm .01)}{(1.4 \pm .1)^2}$$

So,

$$\frac{2(9.99)}{(1.5)^2} \leq g \leq \frac{2(10.01)}{(1.3)^2} \Rightarrow 8.88 \leq g \leq 11.85$$

m/sec²

a quotient of 2 positive numbers (with error bounds) is minimized by taking the smallest numerator & the largest denom!

biggest num
smallest denom

(actual value 9.81 m/sec²)

If we wanted to improve experiment accuracy, what measurement should we improve, e.g. how could we guarantee to get g within $.1$ m/sec²?

$\lim_{x \to c} f(x) = L$ means, roughly speaking, that as x nears c, ($\& \, x \neq c$), f(x) nears L

More precisely, if someone specifies a tolerance (error bound) for how close f(x) should be to L,

then it should be possible to specify an allowable tolerance for how close x should be to c, to guarantee the accuracy in f(x)

(the text has a very good discussion of this.)

Exercise 1 Let $f(x) = 3x+1$

We expect $\lim_{x \to 2} 3x+1 = 7$

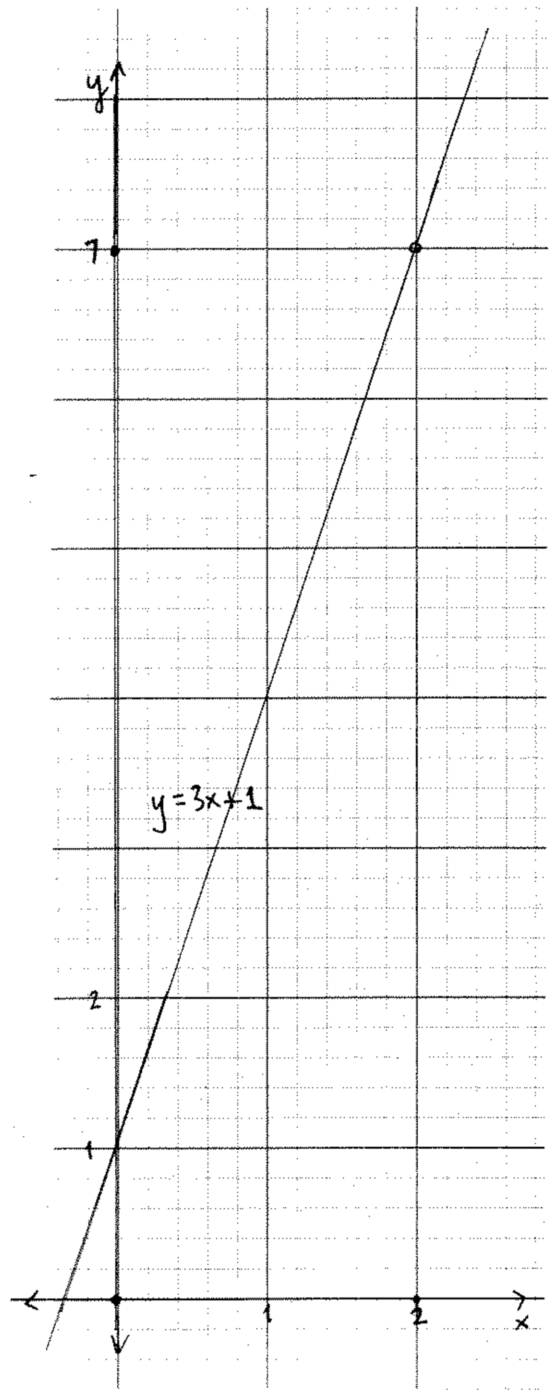
1a) To ensure $|f(x) - 7| < .2$

we should specify

$|x-2| < \boxed{}$?

algebraic:

geometric:



1b) To ensure $|f(x) - 7| < .001$

we should specify

$|x-2| < \boxed{}$?

1c) To ensure $|f(x) - 7| < \varepsilon$ $(\varepsilon > 0)$

pick $|x-2| < \boxed{}$

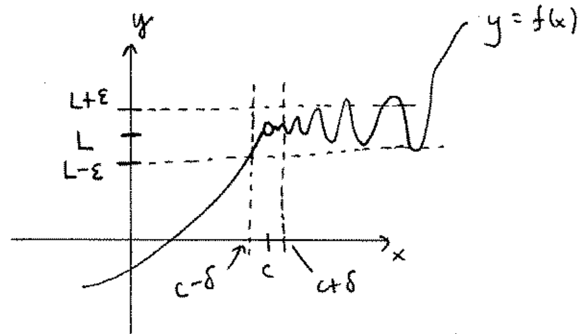
(any smaller positive # would also work)

$\lim_{x \to c} f(x) = L$ means precisely that

for every $\varepsilon > 0$ there exists a $\delta > 0$ so that
"$\forall$" "$\exists$" "s.t."

$|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$

Limit definition



given $\varepsilon > 0$ (error tolerance)
there should exist $\delta > 0$
so the graph of f lies
within the horizontal
strip $L - \varepsilon < y < L + \varepsilon$
for all $0 < |x - c| < \delta$.

Exercise 2: Use the ε-δ def. of limit (and page 2) to show

$$\lim_{x \to 2} 3x + 1 = 7$$

Exercise 3: Let c, m, b be real numbers.

Use the ε-δ def. to prove

(3a) $\lim_{x \to c} b = b$

(3b) For $m \neq 0$, $\lim_{x \to c} mx + b = mc + b$

Exercise 4 Prove

$$\lim_{x \to 2} 3x^2 + 4x = 20$$

proof:

work space

Using ideas like those above, one can prove

Theorem A (page 68) Main Limit Theorem

Let n be a positive integer, k a constant, f and g be functions that have limits at $x = c$

Then

1. $\lim_{x \to c} k = k$ (see 3a)
2. $\lim_{x \to c} x = c$ (see 3b)
3. $\lim_{x \to c} k f(x) = k \lim_{x \to c} f(x)$ (see 4)
4. $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$ (see 4)
5. $\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
6. $\lim_{x \to c} f(x) g(x) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$
7. $\lim_{x \to c} f(x)/g(x) = \lim_{x \to c} f(x) \Big/ \lim_{x \to c} g(x)$ provided $\lim_{x \to c} g(x) \neq 0$
8. $\lim_{x \to c} (f(x))^n = \left(\lim_{x \to c} f(x)\right)^n$
9. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$ provided $\lim_{x \to c} f(x) > 0$ when n is even.