Name Solutions

Student I.D. ______________________

Math 1210-3
Final Exam
April 29, 2003

Please show all work for full credit. When in doubt as to whether your reasoning is clear, add an explanation. This exam is closed book and closed note, ~~but you may use a scientific calculator. You may not use a calculator which does graphing or symbolic differentiation, however.~~ I have provided you with some integral tables, geometry identities, ~~summation and moment formulas.~~ There are 200 points possible, as indicated below and in the exam. You have two hours to complete this exam, so apportion your time accordingly. Good Luck!!

Score

Score	POSSIBLE
1 ______	15
2 ______	25
3 ______	30
4 ______	25
5 ______	15
6 ______	30
7 ______	45
8 ______	15
TOTAL ______	200

1) Compute the following limits

1a)

$$\lim_{x\to 5}\frac{x^2-4x-5}{x^2-6x+5}$$

(5 points)

$$= \lim_{x\to 5}\frac{\cancel{(x-5)}(x+1)}{\cancel{(x-5)}(x-1)} = \frac{6}{4} = \frac{3}{2}$$

1b)

$$\lim_{t\to 1^+}\frac{t+3}{t-t^2}$$

(5 points)

$$= \lim_{t\to 1^+}\frac{t+3}{t(1-t)} = \frac{4}{1(0^-)} = -\infty$$

1c)

$$\lim_{x\to(-\infty)}\frac{3x^3-x^2}{5x^2+2x+x^3}$$

(5 points)

$$= \lim_{x\to -\infty}\frac{\cancel{x^3}\left[3-\frac{1}{x}\right]}{\cancel{x^3}\left[\frac{5}{x}+\frac{2}{x^2}+1\right]} = 3$$

2) Compute the following derivatives

2a)

$$D_x\left(12x^3-5x^5+\frac{61}{x^2}\right)$$

(5 points)

$$= 12\cdot 3x^2 - 25x^4 + 61(-2)x^{-3}$$

$$= 36x^2 - 25x^4 - 122x^{-3}$$

2b)

$$D_t\left(\frac{t^2}{3t+7}\right)$$

(5 points)

$$= \frac{2t(3t+7) - t^2\cdot 3}{(3t+7)^2} = \frac{3t^2+14t}{(3t+7)^2}$$

2c)

$$\left[\frac{d}{dx}\right][\sin(3x+1)]^4$$

(5 points)

$$= 4\left(\sin(3x+1)\right)^3 \cdot \cos(3x+1)\cdot 3$$

2d) Suppose f(1)=5, g(1)=−2, $f'(1)=3$, $g'(1)=2$. Compute the derivative of the function $(f(x)+2g(x))^3$, at $x=1$.

(10 points)

$$D_x\left(f(x)+2g(x)\right)^3 = 3\left(f(x)+2g(x)\right)^2\left(f'(x)+2g'(x)\right)$$

@ $x=1$, $= 3(5-4)^2(3+4) = 21$

3) Compute the following integrals

3a)

$$\int 12\,u^2+3\cos(u)+\frac{21}{u^3}\,du$$

(10 points)

$$= 12\frac{u^3}{3} + 3\sin u + 21\frac{u^{-2}}{-2} + C$$

$$= 4u^3 + 3\sin u - \frac{21}{2}u^{-2} + C$$

3b)

$$\int_0^{\frac{1\pi}{4}} \cos(2\,x)\,[\sin(2\,x)]^2\,dx$$

(10 points)

$x=0$
$u=0$
$x=\pi/4$
$u=1$

$u = \sin 2x$
$du = 2\cos 2x\,dx$

$$\int_0^1 u^2\,\frac{du}{2} = \frac{1}{2}\frac{u^3}{3}\Big]_0^1 = \frac{1}{6}$$

3c)

$$\int_0^2 x\,(x^2+1)^3\,dx$$

(10 points)

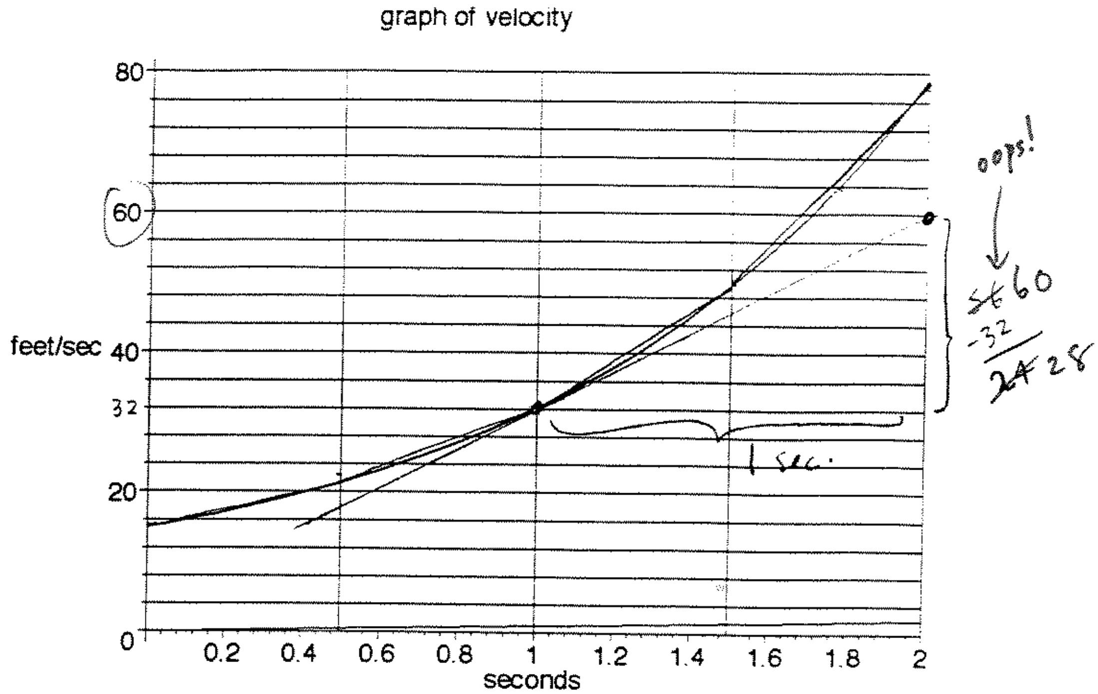
$x=0$
$u=1$
$x=2$
$u=5$

$u = x^2+1$
$du = 2x\,dx$

$$= \int_1^5 u^3\,\frac{du}{2} = \frac{1}{2}\frac{u^4}{4}\Big]_1^5 = \frac{(5^4-1)}{8} = \frac{624}{8}$$

was wrong on earlier version of sol'ns

4) Here is the graph of the velocity function, for an object moving along a straight line:



Use the velocity graph above to answer the following questions. Make sure to use correct units, and to explain your reasoning:

4a) Estimate the object velocity at time t = 1 second.

(5 points)

$v(1) = 32$ ft/sec.

4b) Estimate the acceleration of the object at t=1.

(5 points)

= deriv of vel. = slope of tangent line $\approx \frac{28}{1}$ ft/sec^2

(see graph)

also for 4c):

$\leq \int_0^{.5} v(t)\,dt \leq 10$ $\quad 12 \leq \int_{.5}^{1} v(t)\,dt \leq 14$ $\quad 20 \leq \int_{1}^{1.5} v(t)\,dt \leq 22$ $\quad 30 \leq \int_{1.5}^{2} v(t)\,dt \leq 32$

So $\quad 70 \leq \int_0^2 v(t)\,dt \leq 78$

also, see sol'n on previous page

4c) About how far does the object travel between t=0 and t=2? Obtain an estimate that is accurate to within 8 feet or better. Explain.

(10 points)

$$\int_a^b v(t)\,dt = s(b) - s(a)$$

t	$v(t)$
0	17.5
.5	21
1	32
1.5	50
2	79.

If I use $\Delta t_i = .5$ and trapezoids, I get

$$\int_0^2 v(t)\,dt \approx .5\left(\frac{38.5}{2} + \frac{53}{2} + \frac{82}{2} + \frac{129}{2}\right)$$

$$\approx 75.6 \text{ feet}$$

38.5
53
82
129
$\overline{302.5}$

$302.5 / 4 = 75.625$

Since the max. ht difference between trapezoids & graph is < 2 ft/sec my (slight overestimate) is accurate to within 2 feet/sec · 2 sec = 4 feet

4d) Estimate the average velocity of the object over the time interval [0,2].

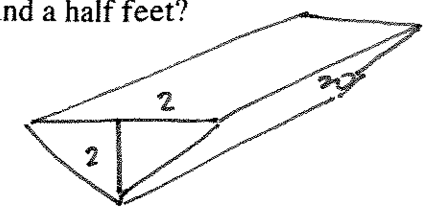
(5 points)

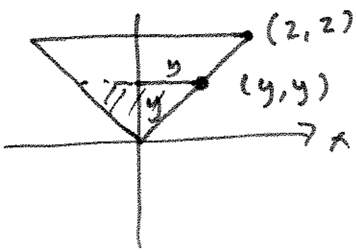
$$V_{av} = \frac{1}{b-a}\int_a^b v(t)\,dt \approx \frac{1}{2}\,75.6 \text{ (from 4c)}$$

$$\approx 37.8 \text{ ft/sec.}$$

5) A drinking trough for cattle has a cross section in the form of an isoceles triangle (with the base at the top, of length 4 feet). The triangle's height is 2 feet and the trough is 20 feet long. If water is filling the trough at the rate of 3 cubic feet per minute, then how fast is the water depth increasing, when the depth is one and a half feet?

(15 points)





$V(t)$ = volume at time t

$y(t)$ = depth at time t

$\frac{dV}{dt} = 3 \quad ft^3/min$

Find $\frac{dy}{dt}$ when $y = 1.5$ ft

$V = 20 \cdot (\text{area of cross-section})$

$= 20 \left(\tfrac{1}{2}\right) \underset{\text{ht}}{y} \underset{\text{base}}{(2y)}$

$V = 20y^2$

$V'(t) = 20 \cdot 2y\, y'(t)$

$= 40\, y\, y'(t)$

at $y = 3/2$: $\quad \cancel{3} = 40\left(\tfrac{3}{2}\right) y'$

$\frac{1}{20} = y'$

$\boxed{y' = \frac{1}{20} \text{ ft/min.}}$

6) Consider the function

$$f(x) = -x^3 + 12x$$

6a) On what intervals is f increasing and decreasing?

(5 points)

$$f'(x) = -3x^2 + 12$$
$$= -3(x^2 - 4)$$
$$= -3(x-2)(x+2)$$

- - - 0 + + + 0 - - -   sign f'

-2   2

f DEC   INC   DEC

6b) On what intervals is f concave up and concave down?

(5 points)

$$f''(x) = -6x$$

+ + + + 0 - - - -   sign f''

0

CU   CD

6c) Identify where all local extrema of f occur, find their values, and identify whether they are local maxima or local minima.

(5 points)

$f(-2) =$ local min

$f(2) =$ local max

6d) Find the x intercepts for the graph of f.

(5 points)

$$0 = x(-x^2 + 12)$$
$$= x(\sqrt{12} - x)(\sqrt{12} + x)$$

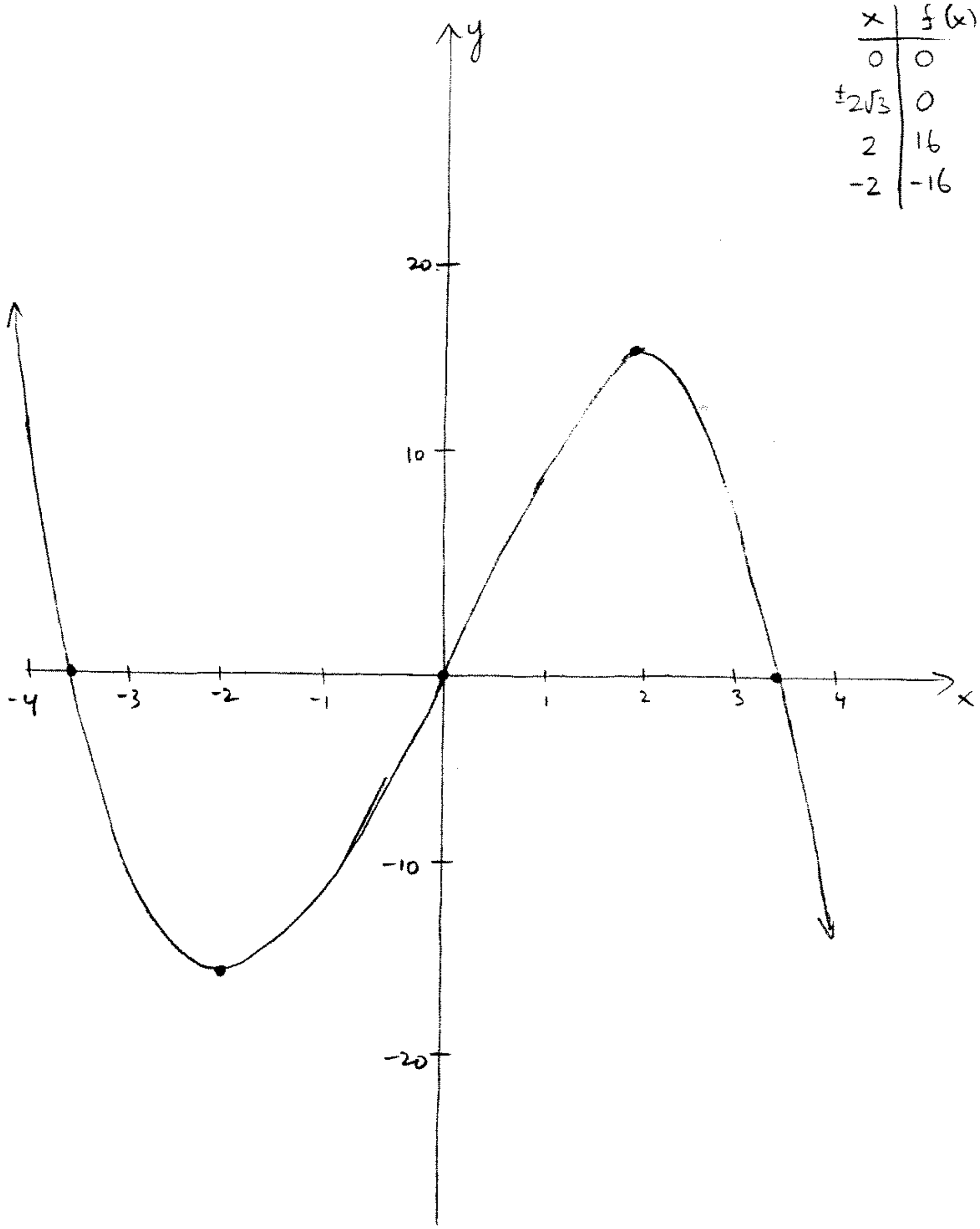
$x = 0, \pm 2\sqrt{3}.$

$f(x) = -x^3 + 12x$

6e) Create an accurate graph for y=f(x), using all of your results from 6a-6d.

(10 points)

x	$f(x)$
0	0
$\pm 2\sqrt{3}$	0
2	16
−2	−16



7) Consider the region bounded between the curves $y=x^2$ and $y=4$.

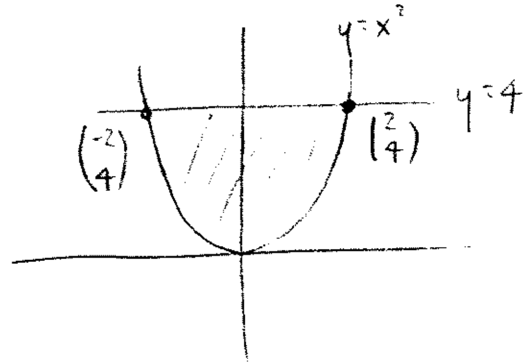
7a) Sketch the region. Find the coordinates where the curves cross.

(5 points)

$x^2 = 4$

$x = \pm 2$

$\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix}$



7b) Find the area of the region

(5 points)

$$A = \int_{-2}^{2} 4 - x^2 \, dx = 2\int_0^2 4 - x^2 \, dx$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2 = 2\left[8 - \frac{8}{3}\right] = \frac{32}{3}$$

7c) Find the centroid of this region.

(15 points)

$$\bar{x} = 0 \text{ by symmetry } \left(M_y = \int_{-2}^{2} x(4-x^2)\,dx = 0 \right)$$

set $\delta = 1$

$$M_x = \frac{1}{2}\int_{-2}^{2} 16 - x^4 \, dx \qquad \left(= \frac{\delta}{2}\int_a^b f(x)^2 - g(x)^2 \, dx \right)$$

$$= \int_0^2 16 - x^4 \, dx$$

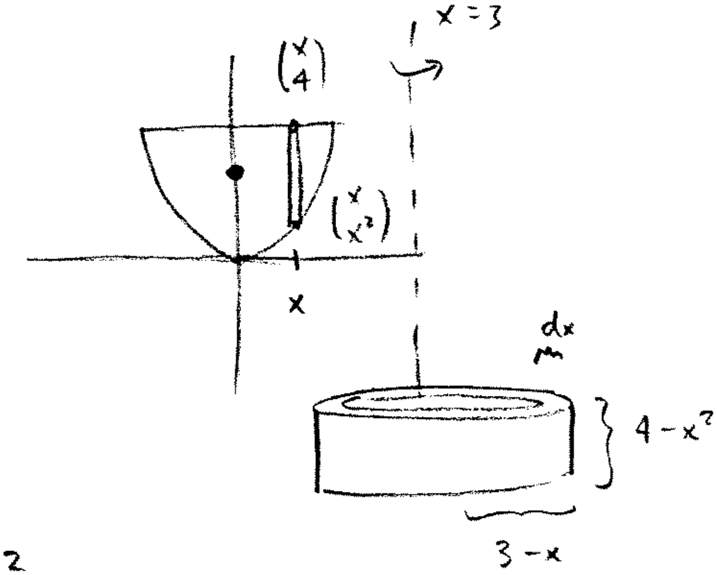
$$= 16x - \frac{x^5}{5}\Big]_0^2 = 32\left(1 - \frac{1}{5}\right) = \frac{128}{5}$$

$$\text{so } \bar{y} = \frac{M_x}{m} = \frac{128}{5} \cdot \frac{3}{32} = \frac{12}{5} = 2.4$$

$$\boxed{\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 2.4 \end{pmatrix}}$$

7d) Use slicing or cylindrical shells to compute the volume of revolution obtained by taking this region and rotating it around the line x=3.

(15 points)



$$dV = 2\pi(3-x)(4-x^2)\,dx$$
$$= 2\pi\left[x^3-3x^2-4x+12\right]dx$$

$$V = \int_{-2}^{2} 2\pi\,(x^3-3x^2-4x+12)\,dx$$

$$= 4\pi\int_0^2 -3x^2+12\,dx$$ (only even terms give $\neq 0$ contribution, since interval is symmetric)

$$= 4\pi\left[-x^3+12x\right]_0^2 = 4\pi\left[-8+24\right] = 64\pi$$

7e) Check your answer in 7d), by using Pappus' Theorem to recompute the volume of revolution.

(5 points)

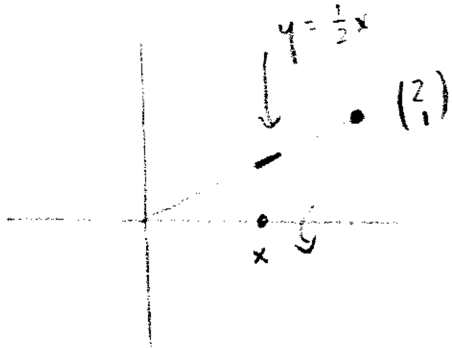
$$V = 2\pi R A \quad = \quad 2\pi\cdot 3\cdot \frac{32}{\cancel{3}} = 64\pi \quad \checkmark$$

R: dist from axis to centroid

A: section area

8a) Consider the line segment $y=\frac{1}{2}x$, for x in the interval $[0, 2]$. Rotate this line segment about the x-axis, to obtain a cone surface. Write down and evaluate the definite integral for this surface's area.

(10 points)



$$A = \int_0^2 \pi x \frac{\sqrt{5}}{2}\, dx = \frac{\sqrt{5}\,\pi}{2} \left.\frac{x^2}{2}\right]_0^2$$

$$\boxed{= \sqrt{5}\,\pi}$$

$$dA = 2\pi y\, ds \qquad ; \quad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi\left(\frac{x}{2}\right)\sqrt{1+\frac{1}{4}}\, dx$$

$$= \pi x \frac{\sqrt{5}}{2}\, dx$$

8b) Check your answer to 8a) by using the area for cone surfaces which you can deduce from your geometry table.

(5 points)

$$A = \pi r \sqrt{r^2+h^2}$$

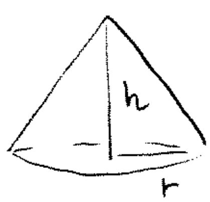
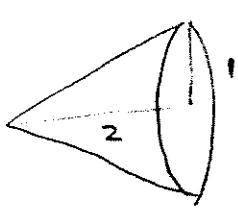


table.

for us



so $A = \pi \cdot 1 \sqrt{1+4} = \boxed{\pi\sqrt{5}}$