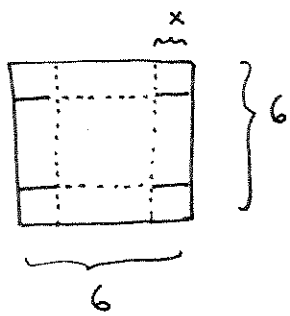
Math 1210-2
Friday Oct 5

§3.1 Maxima and minima

In science, mathematics, business one seeks (or nature seeks) to optimize an outcome or output — mathematically this means that there are functions which are being maximized or minimized

Warmup exercise (for the theory described in §3.1):

Exercise 1 : Your kids want to make baskets of paper squares which are 6 in by 6 in. They will cut and fold according to this diagram:



Find x so that the resulting open-top basket has the most volume possible.
Follow these steps:

1a) Volume $V(x) =$
1b) Physically reasonable domain?
1c) If the maximum value of V occurs at some x inside the domain (i.e. not one of the endpoints), then we expect $V'(x) = 0$ there. Why?
1d) Find the dimensions of the open-top box with the largest volume.

We will use some believable facts (which are proven in more advanced math classes) to organize our work in "max-min" problems.

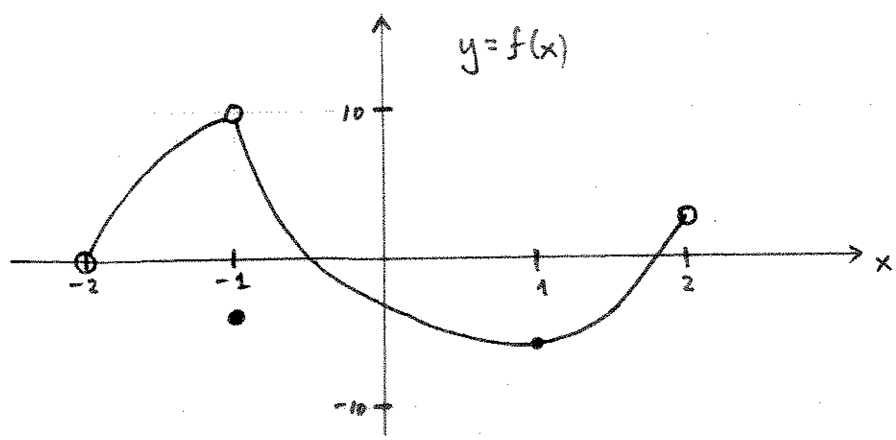
First, definitions:

Let S be the domain of a function f (typically S is an interval)
Let c be a point in S
Then

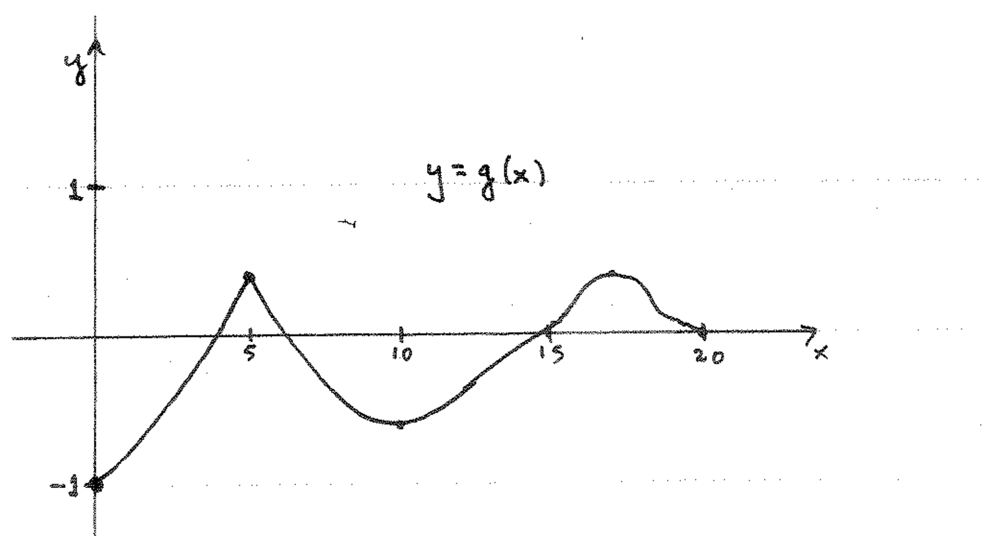
(i) $f(c)$ is a <u>maximum value</u> of f on S, if $f(c) \geq f(x)$ for all x in S
(ii) $f(c)$ is a <u>minimum value</u> of f on S, if $f(c) \leq f(x)$ for all x in S
(iii) $f(c)$ is an <u>extreme value</u> of f on S if it is either a maximum or a minimum value.

<u>Exercise 2</u> From the graphs of functions shown below, and the corresponding domains, find all extreme values, and the points at which they occur.

2a)
$S = (-2, 2)$
i.e. $-2 < x < 2$
an open interval



2b) $S = [0, 20]$
i.e. $0 \leq x \leq 20$
a closed interval and a continuous function



<u>Theorem A</u> If f is continuous on a closed interval $S = [a, b]$, then f attains a maximum and a minimum value on that domain. (e.g. 2b)

<u>Theorem B</u> (says where extreme values can occur)

Let f be defined on an interval I. If $f(c)$ is an extreme value, then either

(i) c is an endpoint of I

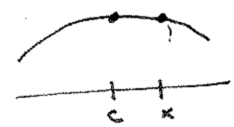
(ii) $f'(c)$ exists an equals 0 (In this case c is called a <u>stationary point</u>)

(iii) $f'(c)$ does not exist (In this case c is called a <u>singular point</u>

<u>proof</u>: Consider the case $f(c)$ is a maximum value.

If c is not an endpoint, and $f'(c)$ does exist, that means we are not in cases (i), (iii).

So let's show $f'(c)=0$:



For $x > c$, $f(x) \leq f(c)$ so $f(x)-f(c) \leq 0$

so $\dfrac{f(x)-f(c)}{x-c} \leq 0$

so $\displaystyle\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$

i.e. $\boxed{f'(c) \leq 0}$

For $x<c$, $f(x) \leq f(c)$ so $f(x)-f(c) \leq 0$

so $\dfrac{f(x)-f(c)}{x-c} \geq 0$ (since $x-c<0$)

so $\displaystyle\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$

i.e. $\boxed{f'(c) \geq 0}$ thus $f'(c)=0$!

We call (i) endpoints

(ii) stationary points

or (iii) singular points

<u>critical points</u>.

By Theorem A & Theorem B, if f is continuous on $[a,b]$, we can find the maximum and minimum values (and where they occur), by checking f at all critical points!

<u>Exercise 3</u> Label all the critical points in exercises 2a), 2b).

Exercise 4 If we choose the domain of $V(x)$ to be $0 \le x \le 3$, in Exercise 1,
what is the maximum value of V?
what is the minimum value of V?

Exercise 5 Let $f(x) = x^{2/3}$ $\left(= (\sqrt[3]{x})^2\right)$. Let the domain of f be the interval $[-1, 2]$.
Find the maximum and minimum values of f on $[-1, 2]$.

Check your work with
this picture of the graph $y = x^{2/3}$