Math 1210-2
Wed October 31

§3.8 Antiderivatives

Recall (?) from first few weeks,

Definition $F(x)$ is an antiderivative of $f(x)$ on the interval $I$ if and only if $D_x F(x) = f(x) \quad \forall x \in I$.

Exercise 1 Find an antiderivative for $f(x) = x^3$ on $(-\infty, \infty)$.

Theorem A (the + C theorem). If $F(x)$ is an antiderivative of $f(x)$ on the interval $I$, then so is $F(x)+C$ (where $C$ is any constant.) Furthermore, every antiderivative $G(x)$ of $f(x)$ can be written

$$G(x) = F(x) + C$$

proof If $D_x F(x) = f(x)$, then $D_x(F(x)+C) = f(x) + 0 = f(x)$ too, so $F(x)+C$ is an antiderivative.

Now, let $G(x)$ also satisfy

$$D_x G(x) = f(x) \qquad \forall x \in I.$$

Then

$$D_x(G(x) - F(x)) = f(x) - f(x) = 0 \qquad \forall x \in I.$$

Write $H(x) := G(x) - F(x)$.

Then $H'(x) = 0 \quad \forall x \in I$.

If we can show $H(x) = C$ is constant, then

$C = G(x) - F(x)$, so $G(x) = F(x) + C$.

Well, pick any $x_0 \in I$, and define $C = H(x_0)$.

For $x \in I$, $x \neq x_0$,

$$\text{MVT} \Longrightarrow \frac{H(x) - \overbrace{H(x_0)}^{C}}{(x - x_0)} = H'(z) = 0, \text{ so } H(x) = C. \quad \blacksquare$$

Definition

$\int f(x)\,dx$ is Leibniz's notation (and ours too), for the set of all antiderivatives of $f(x)$, with respect to the variable $x$

Example: $\int x^3\,dx = \frac{x^4}{4} + C$

Exercise 2 Compute

a) $\int 3x^7 - \frac{4}{x^2} + \frac{5}{\sqrt{x}}\,dx$

b) $\int 4\cos t + 8\sec^2 t\,dt$

c) $\int u^n\,du$ (for $n \neq -1$)

$\int \cos u\,du$

$\int \sin u\,du$

d) $\int f'(x)\,dx =$

e) $D_x\left(\int f(x)\,dx\right) =$

Remark: (d), (e) show that differentiation and antidifferentiation are almost inverse procedures: (e) says that differentiation undoes antidifferentiation. (d) says that (up to an additive constant) antidiff undoes diff.

So, as you might expect, the differentiation rules have corresponding antidifferentiation consequences

<u>Theorem B</u>

(i) $\int k f(x)dx = k \int f(x)dx$ — const mult rule

(ii) $\int f(x)+g(x)\,dx = \int f(x)dx + \int g(x)dx$ — sum rule

(iii) $\int f(g(x))\,g'(x)dx = F(g(x)) + C$ $\left(\begin{array}{c} F(u) \text{ an} \\ \text{antideriv.} \\ \text{of } f(u) \end{array}\right)$ — chain rule!

(iv) $\int f(x)\,g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$ — product rule! (also called "integration by parts")

(v) $\int [g(x)]^r g'(x)dx = \frac{1}{r+1}(g(x))^{r+1} + C$ — generalized power rule; special case of (iii)

Notice, in (iii),(v) if we use differential substitution for

$$u = g(x)$$
$$du = g'(x)dx$$

then they reduce to simpler formulas that we already know:

(iii') $\int f(u)du = F(u) + C$

(v') $\int u^r du = \frac{u^{r+1}}{r+1} + C$

<u>Exercise 3</u> Use (iii) directly, then also with $u=g(x)$, $du = g'(x)dx$ substitution, to find

a) $\int (x^4+3x)^{30}(4x^3+3)dx$

b) $\int \frac{x}{\sqrt{x^2+4}}\,dx$

c) $\int 7\cos 5x \;+\; 3\sin^2(2x)\cos 2x \;dx$