§3.8 Antiderivatives

Recall (2) from first few weeks,

Definition \( F(x) \) is an antiderivative of \( f(x) \) on the interval \( I \)
if and only if \( D_x F(x) = f(x) \) \( \forall x \in I \).

Exercise 1 Find an antiderivative for \( f(x) = x^3 \) on \( (-\infty, \infty) \).

Theorem A (the + C theorem). If \( F(x) \) is an antiderivative of \( f(x) \) on the interval \( I \), then so is \( F(x) + C \) (where \( C \) is any constant.) Furthermore, every antiderivative \( G(x) \) of \( f(x) \) can be written
\[
G(x) = F(x) + C
\]

proof If \( D_x F(x) = f(x) \), then \( D_x (F(x) + C) = f(x) + 0 = f(x) \) too, so \( F(x) + C \) is an antiderivative.

Now, let \( G(x) \) also satisfy
\[
D_x G(x) = f(x) \quad \forall x \in I.
\]

Then
\[
D_x (G(x) - F(x)) = f(x) - f(x) = 0 \quad \forall x \in I.
\]

Write \( H(x) := G(x) - F(x) \).

Then \( H'(x) = 0 \) \( \forall x \in I \).

If we can show \( H(x) = C \) is constant, then
\[
C = G(x) - F(x), \quad \text{so } G(x) = F(x) + C.
\]

Well, pick any \( x_0 \in I \), and define \( C = H(x_0) \).

For \( x \in I, x \neq x_0, \quad \frac{C}{(x-x_0)} = \frac{H(x)-H(x_0)}{x-x_0} = H'(z) = 0, \quad \text{so } H(x) = C. \)
**Definition**

\[ \int f(x) \, dx \] is Leibniz's notation (and ours too), for the set of all antiderivatives of \( f(x) \), with respect to the variable \( x \).

**Example:** \[ \int x^3 \, dx = \frac{x^4}{4} + C \]

**Exercise 2** Compute

a) \[ \int 3x^7 - \frac{4}{x^2} + \frac{5}{\sqrt{x}} \, dx \]

b) \[ \int 4 \cos t + 8 \sec^2 t \, dt \]

c) \[ \int u^n \, du \quad \text{(for } n \neq -1) \]

\[ \int \cos u \, du \]

\[ \int \sin u \, du \]

d) \[ \int f'(x) \, dx = \]

e) \[ D_x \left( \int f(x) \, dx \right) = \]

**Remark:** (d), (e) show that differentiation and antidifferentiation are almost inverse procedures: (e) says that differentiation undoes antidifferentiation, (d) says that (up to an additive constant) antidiff undoes diff.
So, as you might expect, the differentiation rules have corresponding antiderivative consequences.

**Theorem B**

(i) \[ \int k f(x) \, dx = k \int f(x) \, dx \quad \text{const mult rule} \]

(ii) \[ \int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx \quad \text{sum rule} \]

(iii) \[ \int f(g(x)) \, g'(x) \, dx = F(g(x)) + C \quad \text{(antiderivative of } f(u)\text{)} \]

(iv) \[ \int f(x) \, g'(x) \, dx = f(x) \, g(x) - \int f'(x) \, g(x) \, dx \quad \text{chain rule!} \]

(v) \[ \int (g(x))^r \, g'(x) \, dx = \frac{1}{r+1} (g(x))^{r+1} + C \quad \text{product rule!} \]

(also called "integration by parts")

Notice, in (iii),(v), if we use differential substitution for

\[ u = g(x) \]
\[ du = g'(x) \, dx \]

then they reduce to simpler formulas that we already know:

(iii') \[ \int f(u) \, du = F(u) + C \]

(v') \[ \int u^r \, du = \frac{u^{r+1}}{r+1} + C \]

**Exercise 3** Use (iii) directly, then also with \( u = g(x) \), \( du = g'(x) \, dx \) substitution, to find

a) \[ \int (x^4 + 3x) \cdot (4x^3 + 3) \, dx \]
b) \[ \int \frac{x}{\sqrt{x^2 + 4}} \, dx \]

c) \[ \int 7 \cos 5x + 3 \sin^2(2x) \cos 2x \, dx \]