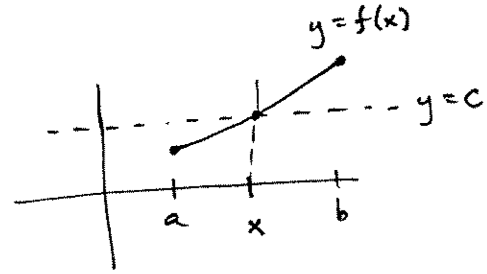
Math 1210-2
Tuesday Oct 30

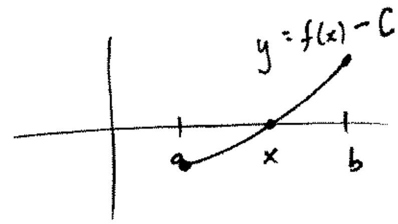
§3.7: Solving equations numerically

means using algorithms to get approximate numerical solutions, especially in cases where exact algebraic solutions cannot be found.

Setup: Suppose we want a solution to $f(x) = C$ and we know
- f is continuous
- $f(a) < C$ and $f(b) > C$ (or $f(a) > C$ and $f(b) < C$)



Step 1: This is the same as finding a root (sol'n) to the equation
$$f(x) - C = 0$$



Bisection algorithm works for any continuous function, and function & interval as above.

$h(x) = f(x) - C$

assume $h(a), h(b)$ have opposite signs

Then repeat the following steps until desired accuracy is obtained:

$x = a$; left guess
$X = b$; right guess
$z = \frac{x+X}{2}$; midpoint of current left and right guesses
if $h(x)h(z) = 0$ then $x = z$ is a root
if $h(x)h(z) > 0$; $\text{sign}(h(x)) = \text{sign}(h(z))$
  set $x = z$ and go to (1)
else set $X = z$ and go to (1)

Advantages: The bisection method works to find (well, approximate) a solution to $f(x) = C$ whenever C is between $f(a)$ and $f(b)$, and f is continuous.

Disadvantage: it's really, really slow.

Example: Approximate $\sqrt{2}$.

As you know (?), $\sqrt{2}$ is irrational!

If it was rational, then we could write

$\sqrt{2} = p/q$ p, q positive integers with no common pos. integer factors (besides 1)

$\Rightarrow 2 = p^2/q^2$

$\Rightarrow 2q^2 = p^2$

Thus p^2 is even, so p is also even.

Thus 2 divides p, so 4 divides p^2, $p^2 = 4m$

$\Rightarrow 2q^2 = 4m$

$\Rightarrow q^2 = 2m$

Thus q^2 is even so q is also even.

Thus 2 divides both p and q.

But we wrote $\sqrt{2} = p/q$ where p, q had no common prime factors.

OOPS! This is an inconsistency, or contradiction.

Thus, it is not possible to write $\sqrt{2} = p/q$, p, q integer.

Exercise 1: To solve $x^2 = 2$ for $x > 0$

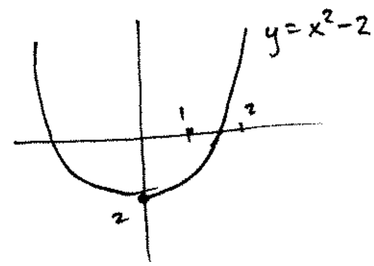
begin with

$[a, b] = [1, 2]$ since $1^2 = 1 < 2 < 4 = 2^2$

$h(x) = x^2 - 2$.

Make a table of successive x, X values (and midpoints z, vals $h(z)$

x	X	$h(x)$	$h(X)$	z	$h(z)$
1	2	-1	$+2$	1.5	.25
1	1.5				



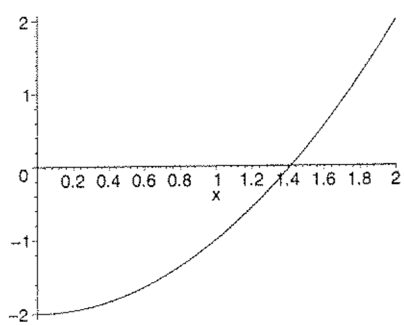
Math 1210-2
October 30, 2007
Bisection and Newton's Methods

Bisection method, applied to approximating the square root of 2:

```
> f:=x->x^2-2;
```

$$f := x \to x^2 - 2$$

```
> with(plots):
> plot(f(x),x=0..2,color=black);
```




```
> x:=1:
  X:=2:  #initial left and right guesses
> for i from 1 to 10 do
    z:=(x+X)/2.0:
      if f(z)*f(x)>=0 then x:=z;
      else X:=z;
      fi;
    print(x,X,f(x),f(X),f(z));
  od:
```

1, 1.500000000, -1, 0.250000000, 0.250000000
1.250000000, 1.500000000, -0.437500000, 0.250000000, -0.437500000
1.375000000, 1.500000000, -0.109375000, 0.250000000, -0.109375000
1.375000000, 1.437500000, -0.109375000, 0.066406250, 0.066406250
1.406250000, 1.437500000, -0.022460938, 0.066406250, -0.022460938
1.406250000, 1.421875000, -0.022460938, 0.021728516, 0.021728516
1.414062500, 1.421875000, -0.000427246, 0.021728516, -0.000427246
1.414062500, 1.417968750, -0.000427246, 0.010635376, 0.010635376
1.414062500, 1.416015625, -0.000427246, 0.005100250, 0.005100250
1.414062500, 1.415039062, -0.000427246, 0.002335547, 0.002335547

So, we see that $1.414062500 < \sqrt{2} < 1.415039062$.

<u>Newton's method</u> uses Calculus to find roots of equations

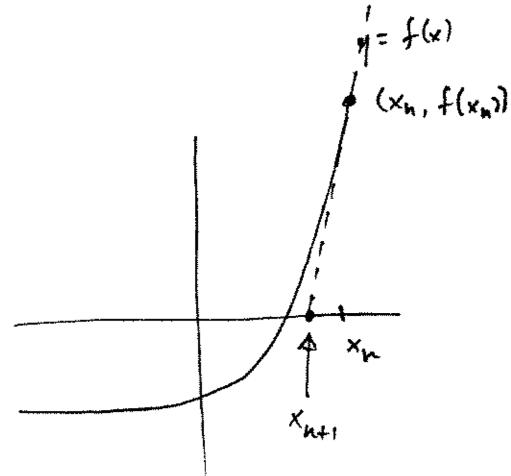
to find a solution x to

$$f(x)=0$$

First find a "good" first guess x_0

Then get x_{n+1} from x_n as follows:

x_{n+1} is the x-value at which the tangent line to $y=f(x)$, passing thru $(x_n, f(x_n))$, intercepts the x-axis.



details: tangent line has slope $f'(x_n)$

so eqtn is

$$y - f(x_n) = f'(x_n)(x - x_n)$$

If we set $y=0$ we get

$$-f(x_n) = f'(x_n)(x - x_n)$$

$$\boxed{x_n - \frac{f(x_n)}{f'(x_n)} = x := x_{n+1}}$$

<u>Advantages</u>: very quickly approximates the root accurately.

<u>Disadvantages</u>: need a good first guess, or the method may not work. Also, requires that $f(x)$ be differentiable.

<u>Exercise 2</u>: Newton to approximate $\sqrt{2}$:

For $f(x) = x^2 - 2$

set $x_0 = 1$.

then $f'(x) = 2x$

$$x_{n+1} = x_n - \frac{(x_n^2 - 2)}{2x_n}$$

$$= \frac{1}{2}x_n + \frac{1}{x_n}$$

Fill in this table

x_0	1
x_1	
x_2	
x_3	

Newton's method applied to approximating $\sqrt{2}$.

```
> Digits:=40:  #number of significant digits
> x:=1.0:   #initial guess
> for i from 1 to 7 do
    x:=x/2+1/x;  #Newton step for our function
    print(x,x^2-2);
  od:
```

$1.500000000000000000000000000000000000000,$

$0.250000000000000000000000000000000000000$

$1.416666666666666666666666666666666666667,$

$0.006944444444444444444444444444444444445$

$1.414215686274509803921568627450980392157,$

$0.600730488273740868896578239138792 8\ 10^{-5}$

$1.414213562374689910626295578890134910117, 0.4510950444942772099280764362\ 10^{-11}$

$1.414213562373095048801689623502530243615, 0.2543584239585437\ 10^{-23}$

$1.414213562373095048801688724209698078570, 0.1\ 10^{-38}$ 6 steps!

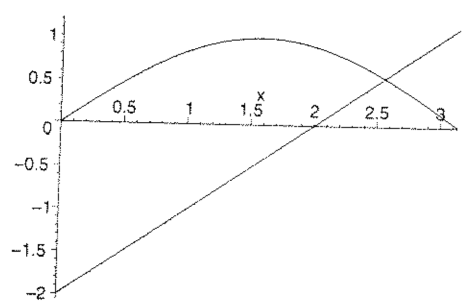
$1.414213562373095048801688724209698078570, 0.1\ 10^{-38}$

```
> sqrt(2.0);  #what Maple thinks the square root of 2 is
```

$1.414213562373095048801688724209698078570$

Newton's method for example 3 page 193: Find where the graphs y=sin(x) and y=x-2 intersect:

```
> restart:
  with(plots):
Warning, the name changecoords has been redefined
> plot({sin(x),x-2},x=0..Pi,color=black);
```




```
> f:=x->sin(x)-x+2;  #we want a root of f
```

$f := x \to \sin(x) - x + 2$

```
> x:=2.5:  #initial guess
> for i from 1 to 6 do
    x:=x-f(x)/(cos(x)-1);

    print(x,f(x));
  od:
```

$2.554672011, -0.000872385$

$2.554195987, -0.63\ 10^{-7}$

$2.554195953, 0.$ 3 steps!

$2.554195953, 0.$

$2.554195953, 0.$

$2.554195953, 0.$

Calculus Carnival

Wednesday, November 14
4:15 PM

Sign up by **Wednesday, November 7** with your calculus instructor or in the Math Center, room 155A

** Extra credit may be awarded, and varies by instructor.
See your instructor for more details **

WIN FABULOUS PRIZES!!