Math 1210-2
Monday Oct 29

Do we want to graph $y = \frac{x}{x^2+1} = f(x)$ from Friday?

§3.6 Mean Value Theorem for derivatives

This section helps explain earlier claims about inc/dec, cu/cD, and is also important for up-coming chapter.

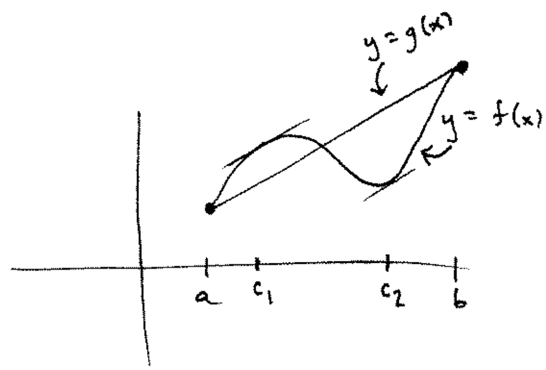
Theorem (MVT). Let f be continuous on the closed interval $[a,b]$. Let f be differentiable on the open interval (a,b).

Then the average rate of change for $a \le x \le b$,

$$\frac{f(b)-f(a)}{b-a}$$

is actually the instantaneous rate of change $f'(c)$, for some $a<c<b$.
(at least one such c exists.)

Precisely: There exists (at least) one c, $a<c<b$ so that $\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$

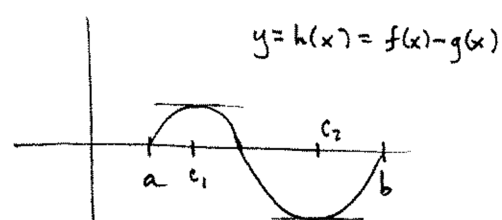


proof. Compare $f(x)$ to the secant line through $(a, f(a))$ & $(b, f(b))$,

i.e. $y - f(a) = \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$

So the secant line is the graph $y=g(x)$, for

$$g(x) = f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$$



Consider $h(x) = f(x) - g(x)$

- h is continuous for $a \le x \le b$
- h is diff'ble for $a<x<b$, $h'(x) = f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right)$.
- $h(a) = h(b) = 0$

If $h(x)$ has a positive max value at $x = c_1$, then $h'(c_1) = 0$
so $f'(c_1) = \frac{f(b)-f(a)}{b-a}$.

If $h(x)$ has a negative min value at $x = c_2$,
then $h'(c_2) = 0$, so
$f'(c_2) = \frac{f(b)-f(a)}{b-a}$

Otherwise $h(x) \equiv 0 \ \forall x \in [a,b]$,
so any $a<c<b$ satisfies the claim $\blacksquare$

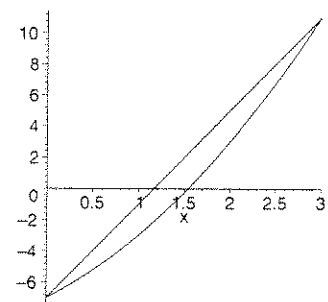
Illustrations of MVT

Exercise 1 : Suppose the highway patrol has timed you on a 40 mile stretch of I15, and it took you ½ hour to travel this distance. Can you claim you were never speeding?

Exercise 2 $f(x) = x^2 + 3x - 7$

$[a,b] = [0,3]$

Find the average rate of change of f, and verify MVT

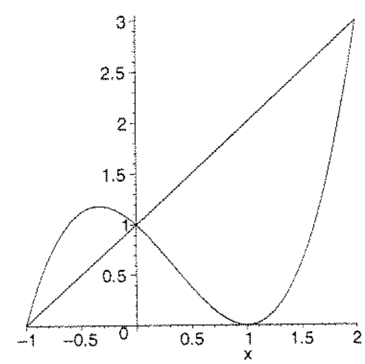


Exercise 3 $f(x) = x^3 - x^2 - x + 1$

$[a,b] = [-1,2]$.

Find average rate of change of f and verify MVT.

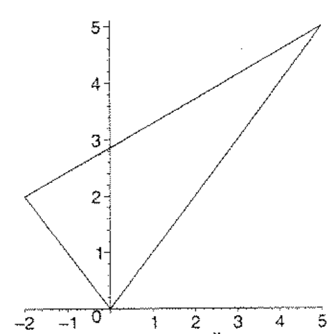
hint: $\frac{1+\sqrt{7}}{3} \approx 1.22$, $\frac{1-\sqrt{7}}{3} \approx -.55$



Exercise 4 $f(x) = |x|$

$[a,b] = [-2,5]$

What is c?



Applications of MVT

① To this chapter:

If f is continuous on $[a,b]$
& differentiable on (a,b)
and if $f'(x) > 0 \quad \forall x \in (a,b)$
then f is <u>increasing</u> on $[a,b]$. (This was half ~~one~~ of our inc/dec theorem).

proof: Let $a \le a_1 < b_1 \le b$

We want to show $f(b_1) > f(a_1)$.

But $\dfrac{f(b_1) - f(a_1)}{b_1 - a_1} = f'(c_1) > 0$. Since $b_1 - a_1 > 0$, this means $f(b_1) - f(a_1) > 0$
i.e. $f(b_1) > f(a_1)$ ∎

(If $f'(x) < 0 \ \forall x \in (a,b)$, the analogous argument shows f is decreasing)

② To the next chapter:

②a If f is continuous on $[a,b]$
& differentiable on (a,b)
and if $f'(x) = 0 \ \forall x \in (a,b)$

Then $f(x)$ is a constant, $f(x) = C \quad \forall x \in (a,b)$.

<u>proof</u> We'll show $f(x) = f(a) \ \forall x \in (a,b)$:

$$\frac{f(x) - f(a)}{x - a} = f'(c) \qquad a < c < x$$
$$= 0, \quad \text{so } f(x) = f(a) \ \blacksquare$$

②b Let $F(x)$ and $G(x)$ be antiderivatives of the same function $h(x)$ on $[a,b]$,
then $G(x) = F(x) + C \quad \forall x \in [a,b]$, where C is a constant

<u>proof</u>: $F'(x) = h(x)$
$G'(x) = h(x)$

so $(G(x) - F(x))' = 0 \quad \forall x \in [a,b]$.

so $G(x) - F(x) = C \quad \forall x \in [a,b]$, i.e. $G(x) = F(x) + C$ ∎
by ②a