Math 1210-2
Friday Nov. 10

§ 4.3 - 4.4 The two Fundamental Theorems of Calculus
(actually, we'll only talk about FTC2 today.)

Recall,

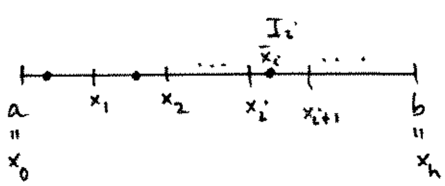
$\int_a^b f(x)\,dx$ is defined to be

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$$

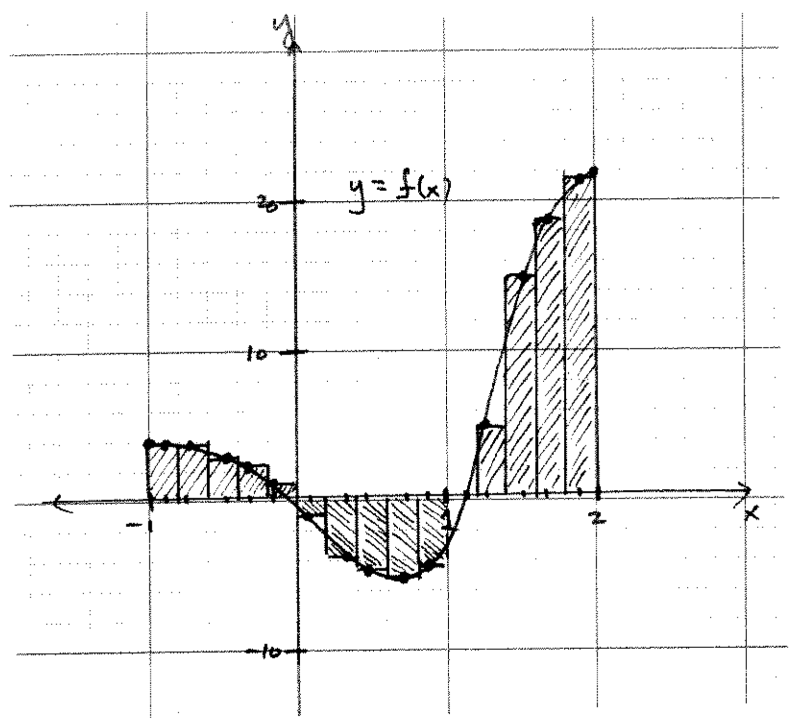
provided the limit exists
(which it does if $f$ is continuous on $[a,b]$).

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i := R_P$$

is called a Riemann Sum,
for the given partition $P$,
and choice of sample points
$\bar{x}_i \in I_i = [x_{i-1}, x_i]$:




Geometric picture of a Riemann sum approximation to $\int_{-1}^{2} f(x)\,dx$




One interpretation of $\int_a^b f(x)\,dx$ is that it equals the sum of the areas of the regions above the x-axis and bounded by the graph $y = f(x)$, minus the areas of the analogous regions below the x-axis

The Second Fundamental Theorem of Calculus, which we shall prove today, says the value of $\int_a^b f(x)\,dx$ can be computed easily if $f(x)$ is continuous and has an antiderivative $F(x)$ on $[a,b]$, via the formula

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \left(\text{which we write as } F(x)\Big]_a^b\right)$$

Exercise 1 : Use geometry to calculate $\int_a^b f(x)$ in the following examples.

When you can, use FTC2 to verify your answer. Sketch region first!

1a) $\displaystyle\int_0^2 5\,dx$

1b) $\displaystyle\int_{-1}^2 3x\,dx$

1c) $\displaystyle\int_{-2}^2 2x+6\,dx$

1d) $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\,dx$

Properties of the definite integral:

(1) $\boxed{\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx}$

$$\| \qquad\qquad\qquad\qquad =$$

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} \left(f(\bar{x}_i)+g(\bar{x}_i)\right)\Delta x_i \qquad\qquad \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i + \lim_{\|P\|\to 0} \sum_{i=1}^{n} g(\bar{x}_i)\Delta x_i$$

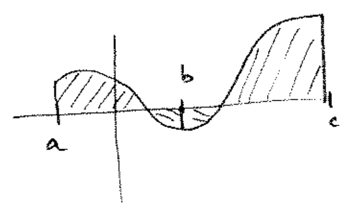
$$= \lim_{\|P\|\to 0}\left(\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i + g(\bar{x}_i)\Delta x_i\right) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i + \sum_{i=1}^{n} g(\bar{x}_i)\Delta x_i$$

(2) $\boxed{\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx} = k \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} k f(\bar{x}_i)\Delta x_i = \lim_{\|P\|\to 0} k \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$$

(3) If $a<b<c$

$\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$




Proof: Let $P$ partition $[a,b]$
& $Q$ partition $[b,c]$
amalgamating $P$ & $Q$ gives a partition $P\#Q$ of $[a,c]$.
Then $(*)$ $R_{P\#Q} = R_P + R_Q$
Take the limit of $(*)$ as $\|P\|, \|Q\| \to 0$, and the result follows.

Exercise 2 Use geometry to show (1c) result, which properties (1)(2) predict:

2a) $$\int_{-2}^{2} 2x+6\,dx = 2\int_{-2}^{2} x\,dx + 6\int_{-2}^{2} 1\,dx$$

2b) $$\int_{-2}^{2} 2x+6\,dx = \int_{-2}^{0} 2x+6\,dx + \int_{0}^{2} 2x+6\,dx$$ (property 3)

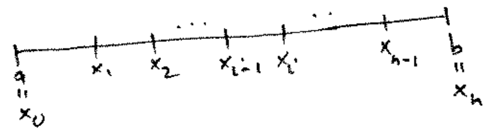
Proof of FTC2:

If $F' = f \ \forall\ a<x<b$, and if $f$ is continuous on $[a,b]$

Then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is slick (and not the proof in our text):

Let $P$ be a partition of $[a,b]$:




$$\begin{aligned}
\text{Then}\quad F(b)-F(a) = \ & F(b) - \cancel{F(x_{n-1})} && = F'(c_n)\Delta x_n && = \sum_{i=1}^{n} f(c_i)\Delta x_i \\
+\ & \cancel{F(x_{n-1})} - \cancel{F(x_{n-2})} && + F'(c_{n-1})\Delta x_{n-1} && \\
+\ & \cancel{F(x_{n-2})} - F(x_{n-3}) && \quad\vdots && \text{since } F'(c_i) = f(c_i). \\
& \quad\vdots && && \\
+\ & \cancel{F(x_i)} - F(x_{i-1}) && + F'(c_i)\Delta x_i && \\
& \quad\vdots && \quad\vdots && \\
+\ & \cancel{F(x_1)} - F(a) && + F'(c_1)\Delta x_1 &&
\end{aligned}$$

↑ apply Mean Value Theorem in each subinterval: there is a $c_i \in I_i$ so that
$$\frac{F(x_i) - F(x_{i-1})}{\Delta x_i} = F'(c_i)$$

Thus, for <u>any</u> partition $P$ it is possible to choose sample points $c_i \in I_i$ so that
$$F(b) - F(a) = R_P$$

(Of course, different sample points could give different values for $R_P$, but we choose these!)

Thus

$$\begin{array}{ccc}
\lim\limits_{\|P\|\to 0} F(b)-F(a) & = & \lim\limits_{\|P\|\to 0} R_P \\
\| & & \| \\
F(b)-F(a) & = & \int_a^b f(x)\,dx
\end{array} \quad \blacksquare$$

<u>Exercise 3</u>: Suppose $f(t)$ is the velocity (ft/sec), of an object with numberline position $F(t)$ ft at time $t$ sec. Interpret the proof above, in this context, to see that it is sensible!