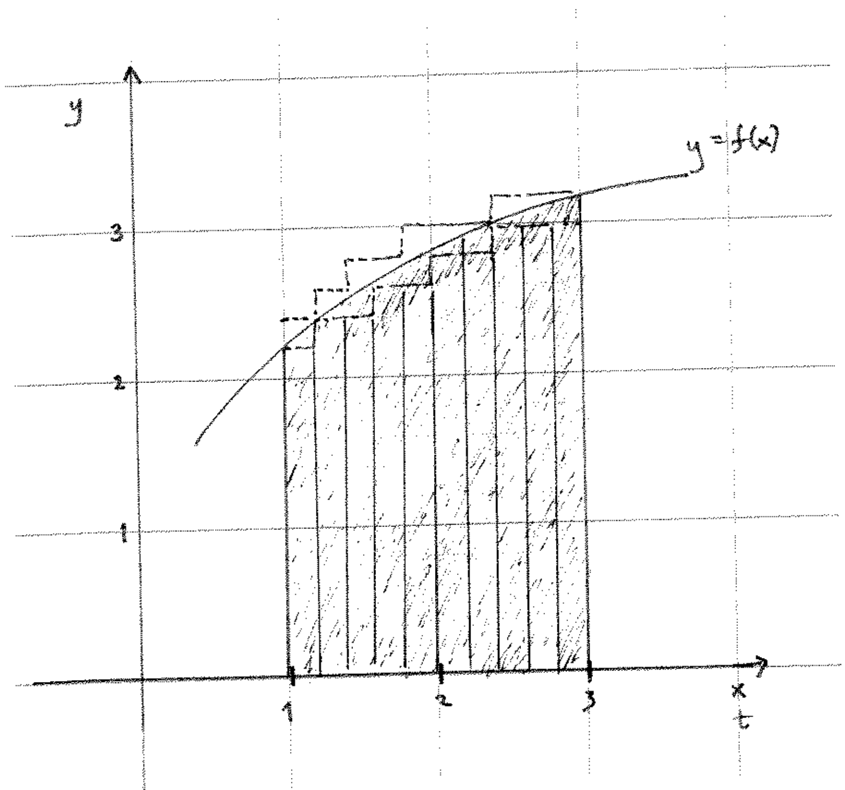
Math 1210-2
Tuesday Nov. 6
Chapter 4: The Definite Integral
§4.1 Introduction to Area

Consider the shaded region below, between the graph $y=f(x)$ and the x-axis,
and with $1 \le x \le 3$.



Exercise 1: Estimate the area above and below, using the indicated tall rectangles of ~~with~~ width $\Delta x = .2$. There are 10 of them.

Notation: Let $a_1, \ldots a_{10}$ be the smaller areas, $(a_1 = (.2)(2.2)$, for example$)$
" $A_1 \ldots A_{10}$ " larger " $(A_1 = (.2)(2.4)$, for example$)$

$$a_1 + a_2 + \cdots + a_{10} = \sum_{i=1}^{10} a_i =$$

$$A_1 + A_2 + \cdots + A_{10} = \sum_{i=1}^{10} A_i =$$

↑
"The sum of A_i,
as i ranges from
1 to 10"

Exercise 2 Same picture as Exercise 1.
Except now, the horizonal axis is t (minutes)
and the vertical axis is y ~~(feet)~~ feet/min
any $y = f(t)$ is the velocity of a turtle walking along a path.
What do your Exercise 1 estimates mean now?

Exercise 3 Same picture as Exercise 1
Same question as Exercise 2.
Except now, horizontal scale is hours
vertical scale is thousands of gallons/hour
and $y = f(t)$ is flow rate of water past a measuring station on a stream

Moral : "Area" can mean different things in applications!

26.8
29.2
5.36
5.84

Summation notation:

$$\sum_{i=1}^{n} b_i = b_1 + b_2 + \dots + b_n$$

Arithmetic:

$$\sum_{i=1}^{n} b_i + c_i = \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} c_i$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} k\, b_i = k \sum_{i=1}^{n} b_i$$

Magic formulas:

$$\sum_{i=1}^{n} i \quad (=1+2+3+\dots+n) = \tfrac{1}{2} n(n+1)$$

__Exercise 4__ Gauss knew that one when he was in 1st grade! How?

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$$\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

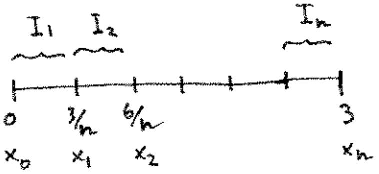
etc.?

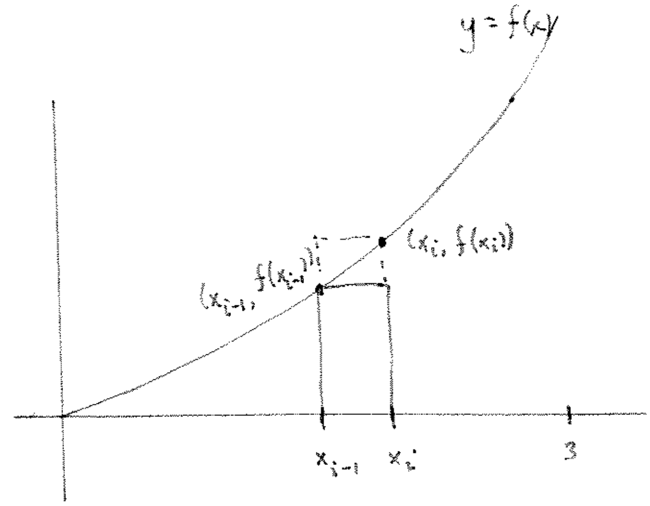
__Exercise 5__: Derive the formula for $\sum_{i=1}^{n} i^2$, by considering $\sum_{i=1}^{n} (i+1)^3 - i^3$!!

Exercise 6 : Find the area under $y = x^2$, $0 \le x \le 3$,

By using n inscribed rectangles (and/or n circumscribed rectangles), of equal widths computing the approximate areas, then letting $n \to \infty$.

(see next page!)





$\Delta x = 3/n$

$= \frac{b-a}{n}$

$I_1 = [0, \Delta x] = [0, 3/n]$

$I_2 = [\Delta x, 2\Delta x] = [3/n, 6/n]$

$I_i = [(i-1)\Delta x, i\Delta x] = \left[\frac{3}{n}(i-1), \frac{3i}{n}\right].$

$= [x_{i-1}, x_i]$

inscribed (within the region) : Since $f(x) = x^2$ is increasing, we want to use fat left endpoints, i.e. $f(x_{i-1})$, for rectangle heights.

$$s_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$=$

circumscribed (around, or outside, the region) Since $f(x) = x^2$ is increasing, use $f(x_i)$ for heights.

$$S_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$=$

$s_n \le A \le S_n$

What happens as $n \to \infty$?

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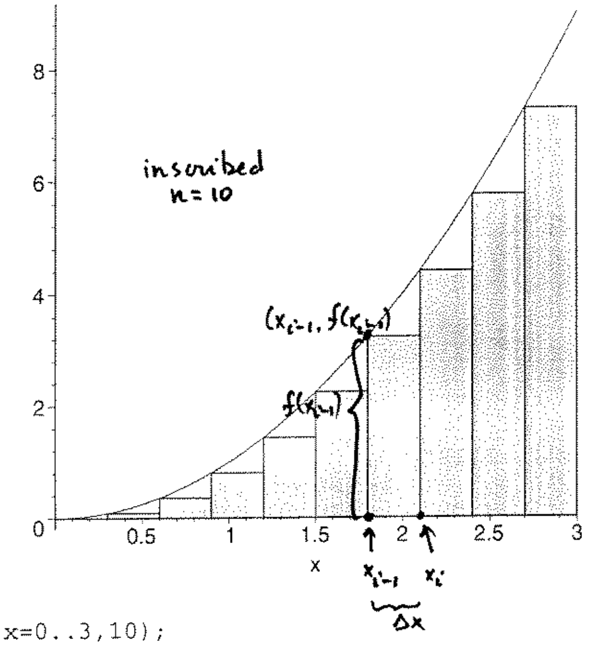
Inscribed and circumscribed rectangle approximations to area

```
> with(plots):
> with(student):
> leftbox(x^2,x=0..3,10);
```



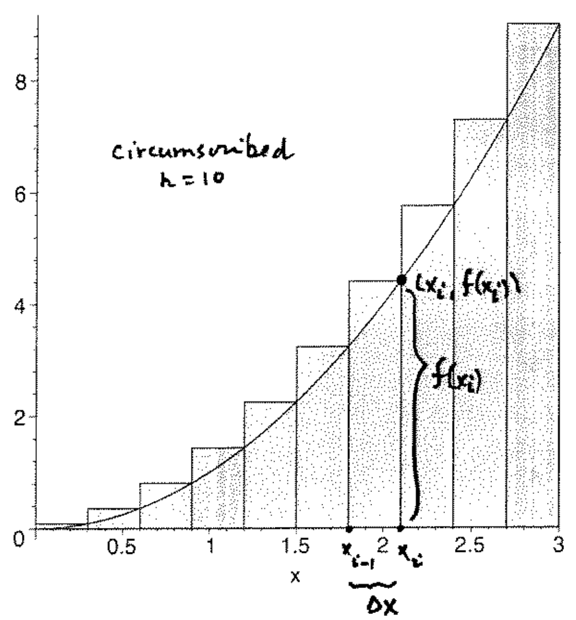

```
> leftsum(x^2,x=0..3,10);
```

$$\frac{3}{10}\left(\sum_{i=0}^{9}\left(\frac{9\,i^2}{100}\right)\right)$$

```
> evalf(%);
```

7.695000000

```
> rightbox(x^2,x=0..3,10);
```




```
> rightsum(x^2,x=0..3,10);
```

$$\frac{3}{10}\left(\sum_{i=1}^{10}\left(\frac{9\,i^2}{100}\right)\right)$$

```
> evalf(%);
```

10.39500000

compare to page 4

```
> evalf(leftsum(x^2,x=0..3,1000));
  evalf(rightsum(x^2,x=0..3,1000));
```

8.986504500
9.013504500

```
>
```