Math 1210-2
Wednesday Nov. 28

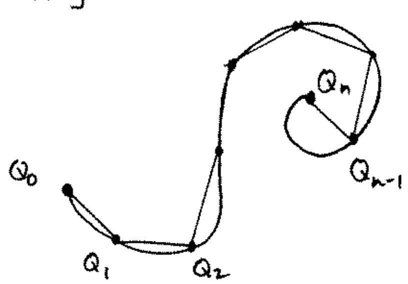
Friday quiz: §5.2-5.3
volumes by slabs and/or
cylindrical shells.

§5.4 Length of a plane curve.

- Before §5.4, recompute volume of radius R ball, using cylindrical shells (Exercise 4 Tuesday)

§5.4
Lengths of curves:

Mathematically (in the absense of string!) the length of a curve is a limit of approximate lengths:



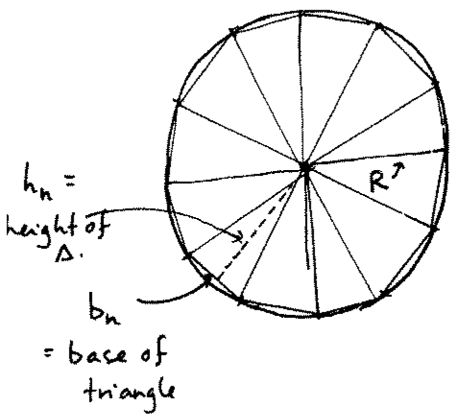
$$\text{Approximate length} = \sum_{i=1}^{n} \text{length}(\overline{Q_{i-1}Q_i})$$

$$\text{Length} = \lim_{\max(\text{length}\,\overline{Q_{i-1}Q_i}) \to 0} \left(\sum_{i=1}^{n} \text{length}(\overline{Q_{i-1}Q_i}) \right)$$

provided the limit exists

Exercise 1 : This is how the Greeks defined π $\left(= \frac{C}{2R}\right)$, and then deduced that the area of a disk is πR^2!

Let C_n be the approximate circumfrence, obtained with an inscribed equilateral n-gon:



$C_n = n b_n$ where b_n is base length of each Δ.

$A_n = n\left(\frac{1}{2} b_n h_n\right)$ h_n = height of each Δ.

A_n is approximate area inside the circle.

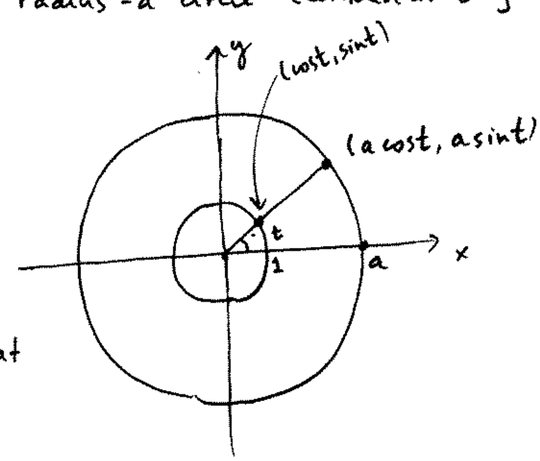
$\underset{\text{actual length}}{\lim_{n\to\infty} C_n} := 2\pi R$ (by scaling, answer must be proportional to R. The Greeks called the proportionality constant 2π)

Bonus: Now take $\lim_{n\to\infty} A_n$ to deduce the area formula.

Parametric curves

Recall (trig.) that the points (x,y) on radius-a circle (centered at origin) are given by

$$x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi$$



in this case "t" is called a parameter

(often we think of t as time, and think of the (x,y) coordinates as the position at time t, of a particle moving along the curve.)

Definition A parametric curve (in the plane), is the collection of points

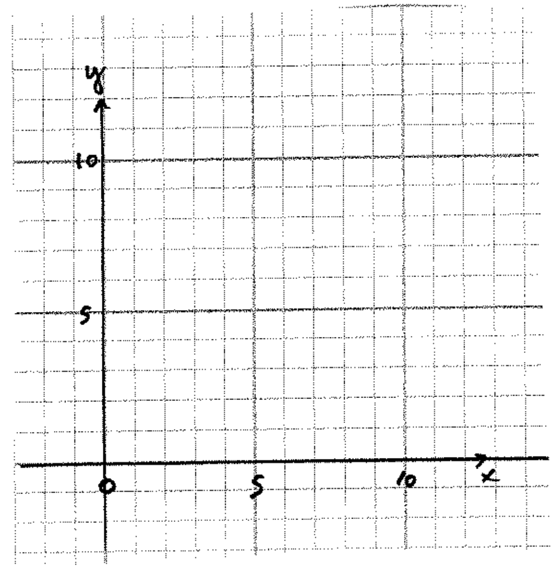
$$x = f(t), \quad y = g(t) \qquad a \le t \le b$$

where f and g are continuous functions

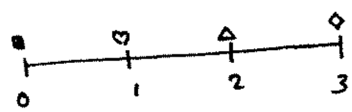
Exercise 2a) Sketch the parametric curve

$$x = 2t + 1 \qquad 0 \le t \le 3$$
$$y = t^2 - 1$$

t	(x,y)
0	
1	
2	
3	



Exercise 2b) Compute the approximate length of the curve in 2a), using the (x,y) points corresponding to this partition of the t-interval $[0,3]$: Get a decimal value!



Exercise 2c) Here is the computer's approximation the the length of the curve, with $n = 30$ equal subdivisions of the t-interval.

```
> f:=t->2*t+1;
  g:=t->t^2-1;
```

$$f := t \rightarrow 2\,t + 1$$
$$g := t \rightarrow t^2 - 1$$

```
> L:=0.0:  #initialize length approximation
> a:=0:
  b:=3:  #t-interval
  n:=30:  #number of subintervals
> h:=(b-a)/n:  #width of each subinterval
> s:=a:  #initial time
> for i from 1 to n do
     L:=L+evalf(sqrt((f(s+h)-f(s))^2+(g(s+h)-g(s))^2));
          #add the length of the next line segment
     s:=s+h:  #update the time parameter
  od:
  L;   #exhibit final length approximation
```

$$11.30448887$$

The following magic integral computes the actual length!

```
> Int(sqrt(4+4*t^2),t=0..3);  #this is the exact value!!
```

$$\int_0^3 2\sqrt{1+t^2}\,dt$$

```
> evalf(%);
```

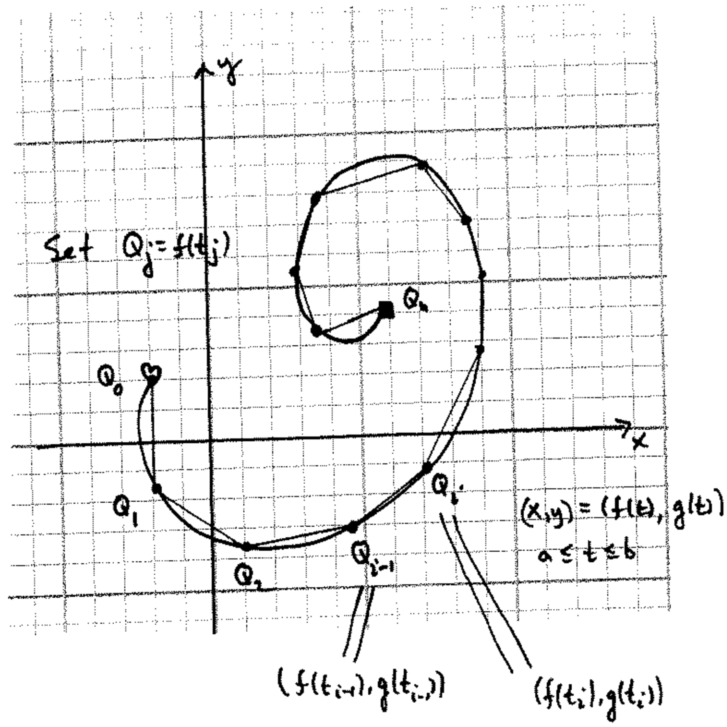
$$11.30527944$$

```
>
```

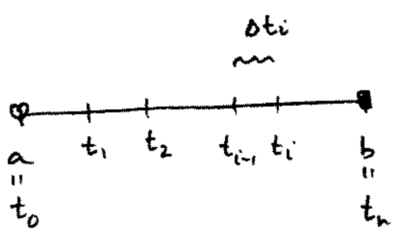
possible numerical computation, because the antiderivative is difficult!

Length of parametric curves via definite integrals:

$x = f(t)$ $\quad a \le t \le b$
$y = g(t)$



Partition $a \le t \le b$:



$$\text{approximate length} = \sum_{i=1}^{n} \text{length}(\overline{Q_{i-1}Q_i})$$

$$= \sum_{i=1}^{n} \sqrt{(f(t_i)-f(t_{i-1}))^2 + (g(t_i)-g(t_{i-1}))^2}$$

$$\frac{f(t_i)-f(t_{i-1})}{\Delta t_i} = f'(\hat{t}_i) \qquad \frac{g(t_i)-g(t_{i-1})}{t_i - t_{i-1}} = g'(\check{t}_i)$$

Mean Value Theorem! (if f and g are differentiable)

$$= \sum_{i=1}^{n} \sqrt{f'(\hat{t}_i)^2 + g'(\check{t}_i)^2}\;\Delta t_i$$

almost a Riemann sum on $a \le t \le b$, for the function $\sqrt{f'(t)^2+g'(t)^2}$ (except used two sample points at once)

If f' & g' are continuous on $[a,b]$, then

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} \sqrt{f'(\hat{t}_i)^2+g'(\check{t}_i)^2}\;\Delta t_i = \boxed{\int_a^b \sqrt{f'(t)^2+g'(t)^2}\,dt := \text{Length}}$$

Exercise 3 Verify that the definite integral for length gives the correct length for

Exercise 1 $x = a\cos t$ $\quad 0 \le t \le 2\pi$
$y = a\sin t$

Exercise 2 $x = 2t+1$ $\quad 0 \le t \le 3$
$y = t^2 - 1$