Math 1210

Tuesday Nov 20

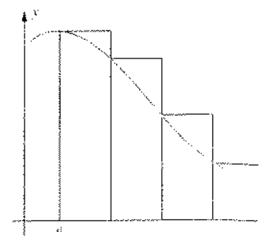
§ 4.6 Numerical Integration : How to approximate definite integrals if antidifferentiation is not available: Text, page 262

1. Left Riemann Sum

Area of ith rectangle $= f(x_{i-1})\,\Delta x_i = \frac{b-a}{n} f\left(a + (i-1)\frac{b-a}{n}\right)$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right)$$

$E_n = \frac{(b-a)^2}{2n} f'(c)$ for some c in $[a, b]$

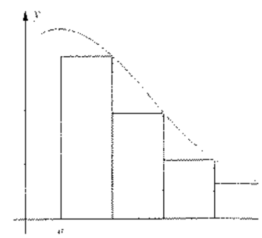


2. Right Riemann Sum

Area of ith rectangle $= f(x_i)\,\Delta x_i = \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right)$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

$E_n = -\frac{(b-a)^2}{2n} f'(c)$ for some c in $[a, b]$

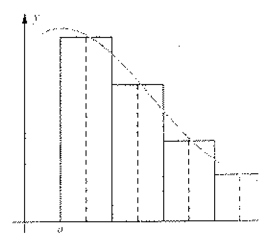


3. Midpoint Riemann Sum

Area of ith rectangle $= f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i = \frac{b-a}{n} f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{n}\right)$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{n}\right)$$

$E_n = \frac{(b-a)^3}{24n^2} f''(c)$ for some c in $[a, b]$



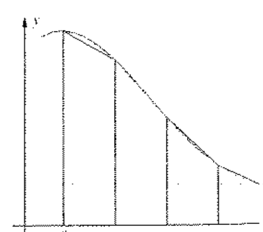
4. Trapezoidal Rule

Area of ith trapezoid $= \frac{b-a}{n}\,\frac{f(x_{i-1}) + f(x_i)}{2}$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}$$

$$= \frac{b-a}{2n}\left[f(a) + 2\sum_{i=1}^{n-1} f\left(a + i\frac{b-a}{n}\right) + f(b)\right]$$

$E_n = -\frac{(b-a)^3}{12n^2} f''(c)$ for some c in $[a, b]$

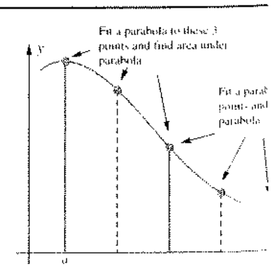


5. Parabolic Rule (n must be even)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

$$= \frac{b-a}{3n}\left[f(a) + 4\sum_{i=1}^{n/2} f\left(a + (2i-1)\frac{b-a}{n}\right) + 2\sum_{i=1}^{n/2-1} f\left(a + 2i\frac{b-a}{n}\right) + f(b)\right]$$

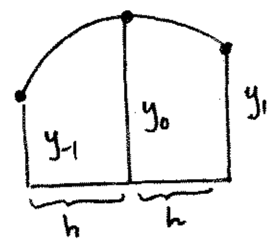
$E_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(c)$ for some c in $[a, b]$



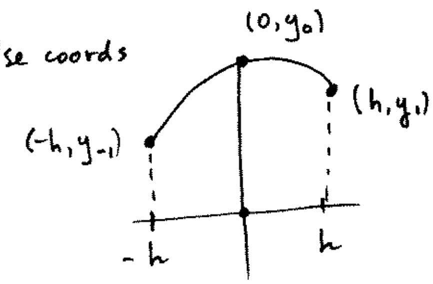
Exercise 1 Explain Trapezoid Rule. How is it related to the Left and Right Riemann Sums?

<u>Exercise 2</u> Simpson's Parabolic Rule is based on the area under parabolas, precisely the fact that if $f(x)$ is a quadratic function interpolating the three points below, then the signed area (definite integral) is given by

$$A = \frac{h}{3}\left(y_{-1} + 4y_0 + y_1\right)$$



Use coords



to find the parabola $p(x) = ax^2 + bx + c$ so that $(-h, y_{-1}), (0, y_0), (h, y_1)$ are on the graph.

then find $\int_{-h}^{h} p(x)\,dx$, show it equals $\frac{h}{3}(y_{-1} + 4y_0 + y_1)$.

Math 1210-2
Numerical Integration

Here are two little subroutines to approximate integrals using the Trapezoid and Parabolic Rules

```
> Trapezoid:=proc(f,a,b,n)
    #f is the function, a,b are the endpoints,
    #n is the number of subintervals
  local
     Trap,  #current Trapezoid appprox
     x,     #current x-value
     dx,    #subinterval width
     i;     #index

  Trap:=evalf(f(a)+f(b)): #initialize
  x:=a:   #start at left endpoint
  dx:=evalf((b-a)/n);  #subinterval length
  for i from 1 to n-1 do
    x:=x+dx:
    Trap:=Trap+2*f(x):
  od:
  Trap:=Trap*dx/2:
  return(Trap);
  end:
```

```
> Simpson:=proc(f,a,b,n)
    #f is the function, a,b are the endpoints,
    #n is the number of (even) subintervals
  local
     Simp,  #current Simpson appprox
     x,     #current x-value
     dx,    #subinterval width
     i;     #index

  Simp:=0: #initialize
  x:=a:   #start at left endpoint
  dx:=evalf((b-a)/n);  #subinterval length
  for i from 0 to n-2 by 2 do
    Simp:=Simp+f(a+i*dx)+4*f(a+(i+1)*dx)+f(a+(i+2)*dx):
  od:
  Simp:=Simp*dx/3:
  return(Simp);
  end:
```

Let's test:
Exercise 3) By hand, work out the Trapezoid and Simpson Approximation to

$$\int_0^1 x^2\, dx$$

with n=4 subdivisions

Compare to

```
> f:=x->x^2;
```

$$f := x \to x^2$$

```
> Trapezoid(f,0,1,4);
  Simpson(f,0,1,4);
```

0.3437500000
0.3333333333

Why is Simpson exact on this example?

```
> Simpson(f,0,1,2);
```

0.3333333333

Exercise 4: Estimate

$$\int_1^2 \frac{1}{t}\,dt$$

Using n=2, Trapezoid and Simpson. The exact value is the natural logarithm of 2, since

$$\ln(x) := \int_1^x \frac{1}{t}\,dt$$

Compare hand work to MAPLE:

```
> g:=x->1/x:
  Trapezoid(g,1,2,2);
  Simpson(g,1,2,2);
  ln(2.0);
  Trapezoid(g,1,2,10);
  Simpson(g,1,2,10);
```

0.7083333330
0.6944444447
0.6931471806
0.6937714030
0.6931502310

~~Exercise 3~~: Estimate π.

```
> h:=x->sqrt(1-x^2);
```

$$h := x \to \sqrt{1-x^2}$$

```
> 4*Simpson(h,0,1,10000); #This should approximate Pi
  evalf(Pi);  #Maple's approximation
```

3.141592177
3.141592654

Exercise 5 Set up #29 p.270

29. Figure 8 shows the depth in feet of the water in a river measured at 20-foot intervals across the width of the river. If the river flows at 4 miles per hour, how much water (in cubic feet) flows past the place where these measurements were taken in one day? Use the Parabolic Rule.

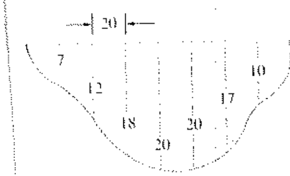


Figure 8